
Analytical solution proposal to vehicle dynamic handling properties

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Abstract: One very important aspect of the vehicle handling behaviour is how it reacts to dynamic inputs from the driver. While much has been done in the analytical realm to describe the vehicle steady state handling properties, the transient and dynamic behaviour of the vehicle have been studied mostly through multibody software packages, as the analytical solution is usually more difficult to be achieved. Taking into consideration that the analytical approach provides the engineer a deeper understanding of the underlying physical phenomena being studied, bringing more simplicity to the overall solution at the same time, this paper proposes an analytical solution for the vehicle lateral acceleration response to periodic excitation at the steering wheel. In the sequence, a comparison of the results between this analytical model and a detailed multibody model is performed, showing that the proposed analytical model is capable of reproducing the more detailed multibody model with good accuracy.

Keywords: handling; analytical solution; frequency response; simulation; vehicle dynamics.

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1 Introduction

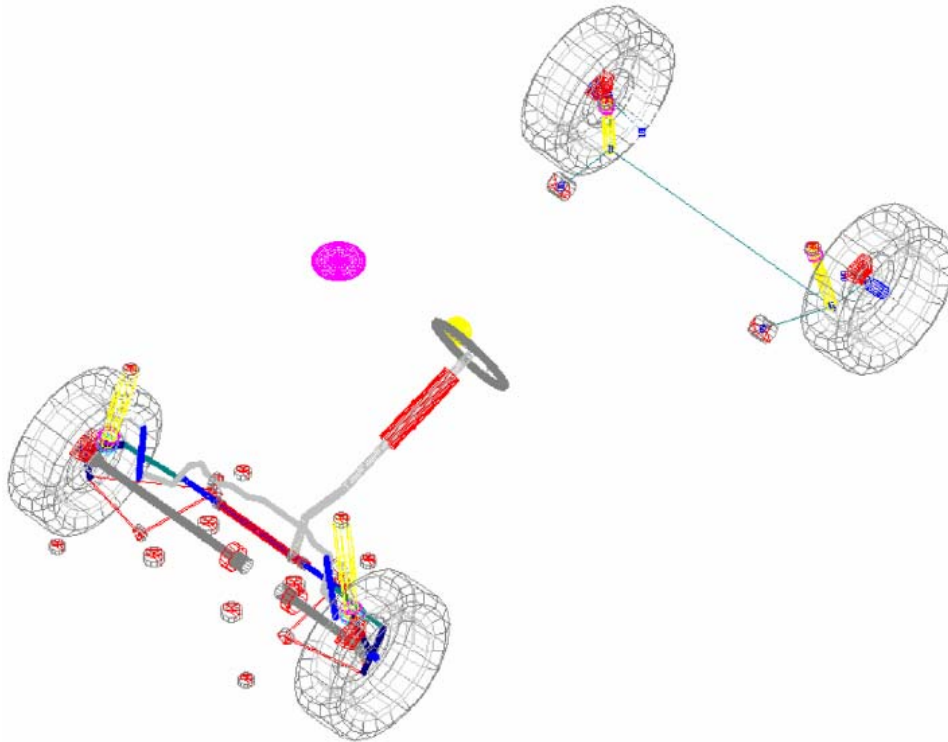
Traditional literature for vehicle dynamics, like Milliken and Milliken (1995), Wong (2001) and Pacejka (2002), covers in detail the analytical description of the vehicle handling for steady state measures, defining very important metrics like understeer gradient, steering sensitivity and roll gradient among others. The analytical solutions to these metrics are very important tools to the development engineer that is able to have a very good understanding of the underlying phenomena and how the tuning variables affect each of these metrics. Besides that, the analytical solutions are extremely efficient in terms of computation time, allowing their usage for quick studies and very early assessments, as well as their linkage to numerical optimisation processes that take the advantage of their computational efficiency.

If for one side there is no question that these steady state parameters are very good indicators of vehicle handling performance and of fundamental importance for vehicle development in this regard, on the other hand, they are not complete, in the sense that they do not capture dynamic variations with the frequency of the inputs by the driver and the transients before achieving the steady state condition.

To be able to simulate these dynamic effects, the engineer usually considers the usage of multibody software packages, as it has been done using ADAMS[®] in previous works by Vilela (2001) and Prado et al. (2001). By doing so, the engineer can get very accurate results for these dynamics conditions through detailed models of the vehicle and a common multibody model easily contains more than 100 degrees of freedom (see example in Figure 1). The main drawback of this approach is that the more the multibody model gets details in the vehicle construction representation, the more difficult and less intuitive is for the engineer the understanding of the basic dynamic phenomena being studied. Besides that, as these models usually contain lots of details in their construction, it is more difficult for the engineer to correctly guess which of the tuning variables affects more the metric of interest. Finally, the computational running time of such models is not as efficient as an analytical solution and, while this might not be a big problem for the normal development cycle in the industry with the current computing capabilities available, it might become a bottleneck for numerical optimisation procedures that demand a very high number of iterations to get to an optimum design.

For the dynamic handling performance, a very important class of metrics is related to how the steady state responses of the vehicle vary as a function of the frequency of steering wheel input by the driver. Kunkel and Leffert (1988) demonstrate how this class of parameters is objectively evaluated through the so-called ‘frequency response test’ where, among other metrics, the variation of the steering sensitivity (lateral acceleration response of the vehicle) with the steering wheel input frequency is evaluated. This same characteristic is evaluated through a detailed multibody model by Prado et al. (2001) for a passenger bus.

Figure 1 Graphical representation of a detailed multibody model (ADAMS[®]) (see online version for colours)



Understanding that this is an important dynamic characteristic of the vehicle that is not satisfactorily studied through the analytical approach in the literature, the purpose of this paper is to propose an analytical model for the vehicle lateral acceleration response to periodic excitation at the steering wheel by the driver, comparing the results with a much more detailed multibody model, so as to help in quantifying the accuracy of the results of the proposed analytical model.

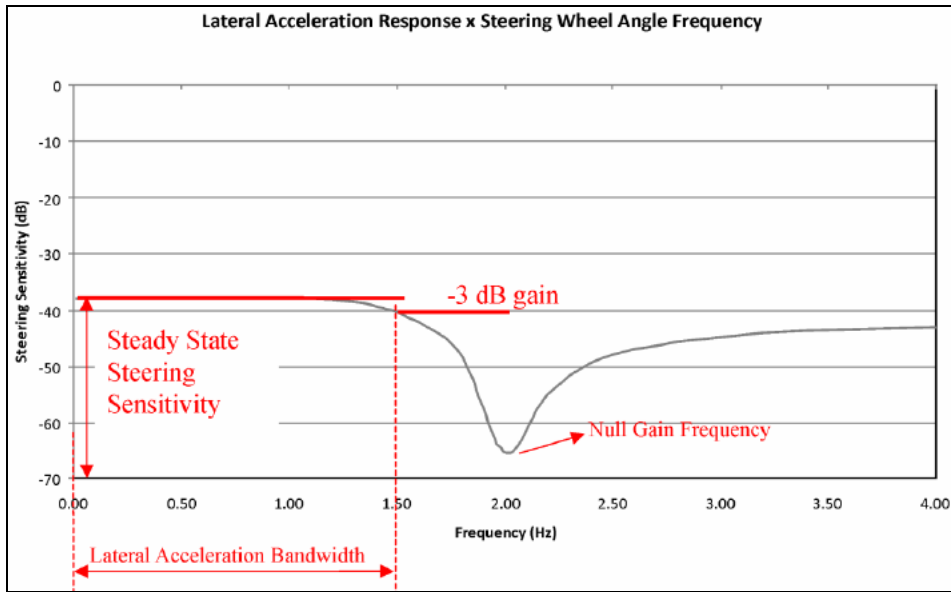
The analytical solution herein proposed is analogous to a simple angular mass-spring-damper system, allowing to the engineer the adoption of other analogies from this simple mass-spring-damper system resolution that can be very helpful during the vehicle development phase. In this sense, this analogy allows the engineer to study the response of the vehicle for inputs other than sinusoidal, like step steer inputs or impulse inputs.

2 Lateral acceleration response metrics for periodic excitation

When evaluating the lateral acceleration response of the vehicle varying the excitation frequency of the steering wheel (harmonic response to a sinusoidal excitation type), it can be noticed that this lateral acceleration response to the steering wheel input starts with a decreasing behaviour in regard to the excitation input frequency, eventually achieving a minimum response value for a specific frequency, which will be used here as a metric

and named as *null gain frequency*, as the gain value at this frequency is very close to zero (it is indeed null for the analytical model proposed, as will be demonstrated further). Besides this metric, this work will consider the *lateral acceleration bandwidth*, which is defined as the frequency value at which there is a reduction in the lateral acceleration response that is noticed by most users. Here, the definition of 3 dB gain reduction proposed by Kunkle and Leffert (1988) will be adopted. Figure 2 shows these definitions at the lateral acceleration response graphic.

Figure 2 Definition of lateral acceleration bandwidth and null gain frequency (see online version for colours)



3 Analytical formulation of the lateral acceleration response to periodic excitation

The objective of this chapter is to present the deduction of the analytical equations for the lateral acceleration response to periodic excitation. Some simplifications are introduced to allow the usage of analytical expressions – the simplifications are explained, and their overall validation verified by the comparison with the results of a detailed multibody model. Figure 3 and Figure 4 show the sketch of the vehicle on a curved path in the XY (top view) and YZ (front view) planes respectively:

The application of Newton's second law to the lateral direction implies in the following:

$$Ma_y = \sum F_y + MR\Omega^2 + 2M\dot{\theta}\dot{z} - 2M\Omega\dot{x} \quad (1)$$

In order to simplify the analytical expression obtained from (1), the following points were considered:

- the vehicle longitudinal velocity \dot{x} is a constant value V_x
- \dot{z} is only a function of the roll angle θ (consideration of flat road without irregularities)
- for the range of interest of the expression obtained (low lateral acceleration limits – from steady state until -0.4 g), θ has a small absolute value, so that $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx \theta$
- the higher order terms in β , θ e Ω are neglected, what is a reasonable assumption as these higher order terms tend to have a much smaller influence compared to the main terms (this simplification will be checked along with the others in the comparison with the detailed multibody model results).

The equation (1) can be then re-written as:

$$\sum F_y = Ma_L \quad (2)$$

$$\text{with } a_L = \frac{V_x^2}{R} \quad (3)$$

Differently from the steady state condition, the derivative of the sideslip angle β is not null. Considering the relative angular velocity $\dot{\beta}$ and the drag angular velocity Ω , for a circular movement with instantaneous radius R , as presented in Figure 5, the kinematic of a rigid body provides the following absolute velocity:

$$V_x = R(\dot{\beta} + \Omega) \text{ or } \frac{1}{R} = \frac{(\dot{\beta} + \Omega)}{V_x} \quad (4)$$

The lateral acceleration a_L can then be re-written:

$$a_L = \frac{V_x^2}{R} = V_x^2 \frac{(\dot{\beta} + \Omega)}{V_x} = V_x (\dot{\beta} + \Omega) \quad (5)$$

Figure 5 shows the vehicle bicycle model with the consideration of the derivative of the sideslip angle β as shown in equation (4).

Figure 3 Vehicle on a curved path – top view (see online version for colours)

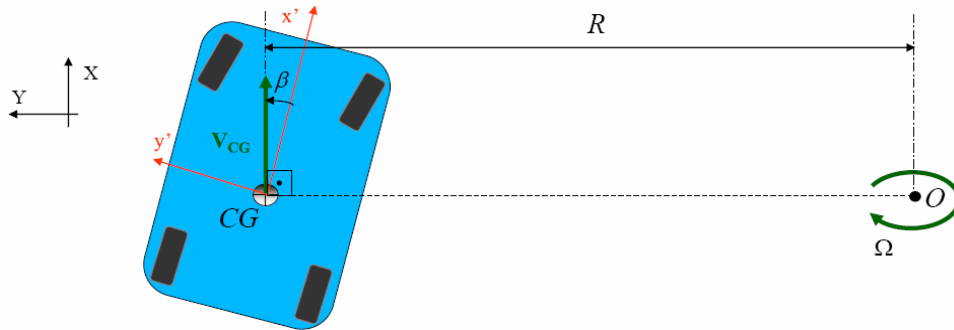


Figure 4 Vehicle on a curved path – front view (see online version for colours)

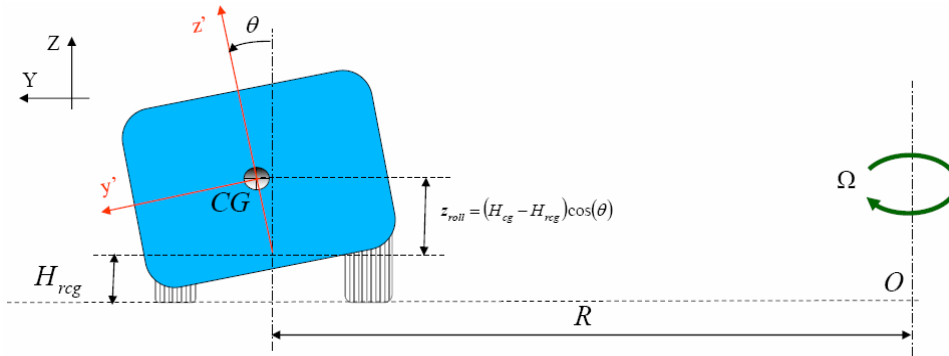


Figure 5 Vehicle bicycle model (see online version for colours)

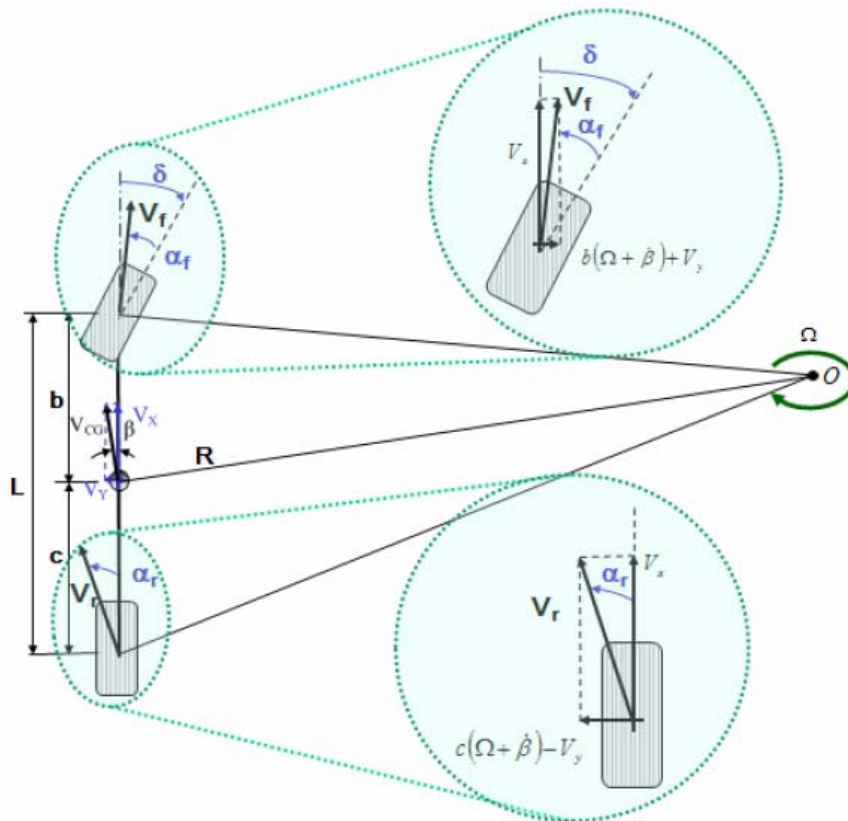


Figure 5 can be applied to determine the front and rear tyre slip angles for a general situation (non-steady state), as long as the curve radius R is considered to be varying with time (not constant). Applying the results from (4) and (5), the slip angles are then calculated:

$$\alpha_f = \delta - \frac{b(\dot{\beta} + \Omega) + V_y}{V_x} \rightarrow \alpha_f = -\beta + \delta - \frac{b(\dot{\beta} + \Omega)}{V_x} \rightarrow \alpha_f = -\beta + \delta - \frac{b}{V_x^2} a_L \quad (6)$$

$$\alpha_r = \frac{c(\dot{\beta} + \Omega) + V_y}{V_x} \rightarrow \alpha_r = -\beta + \frac{c(\dot{\beta} + \Omega)}{V_x} \rightarrow \alpha_r = -\beta + \frac{c}{V_x^2} a_L \quad (7)$$

For low lateral acceleration levels (range of interest for the final analytical expression), it is reasonable to assume that the tyres are working within their linear range, so that the external lateral force produced by them is proportional to the tyre cornering stiffness C_{α} as follows (2 tyres per axle):

$$F_{ext,yf} = 2C_{\alpha f} \alpha_f = 2C_{\alpha f} \left(-\beta + \delta - \frac{b}{V_x^2} a_L \right) \quad (8)$$

$$F_{ext,yr} = 2C_{\alpha r} \alpha_r = 2C_{\alpha r} \left(-\beta + \frac{c}{V_x^2} a_L \right) \quad (9)$$

$$Ma_L = 2C_{\alpha f} \left(-\beta + \delta - \frac{b}{V_x^2} a_L \right) + 2C_{\alpha r} \left(-\beta + \frac{c}{V_x^2} a_L \right) \quad (10)$$

It is possible then to define auxiliary terms A_1 and A_2 , isolating the sideslip angle β in the sequence:

$$A_1 = M + \frac{2}{V_x^2} (C_{\alpha f} b - C_{\alpha r} c) \quad (11)$$

$$A_2 = 2(C_{\alpha f} + C_{\alpha r}) \quad (12)$$

$$A_1 a_L + A_2 \beta = 2C_{\alpha f} \delta \rightarrow \beta = \frac{2C_{\alpha f} \delta - A_1 a_L}{A_2} \quad (13)$$

The equation (5) can have the angular velocity Ω isolated and its derivative with respect to time calculated in the sequence:

$$\Omega = \frac{a_L}{V_x} - \beta \rightarrow \dot{\Omega} = \frac{\dot{a}_L}{V_x} - \dot{\beta} \quad (14)$$

The application of the angular momentum theorem in conjunction with the equation (14) leads to the following:

$$\sum M_{CG,z}^{ext} = J_z \dot{\Omega} \rightarrow J_z \left(\frac{\dot{a}_L}{V_x} - \ddot{\beta} \right) = bF_{yf} - cF_{yr} \quad (15)$$

Applying the results from (8) and (9) in (15):

$$J_z \left(\frac{\dot{a}_L}{V_x} - \ddot{\beta} \right) = 2(cC_{ar} - bC_{af})\beta - \frac{2}{V_x^2} (b^2C_{af} - c^2C_{ar})a_L + 2bC_{af}\delta \quad (16)$$

Once more, it is possible to define auxiliary terms A_3 and A_4 to help with the solution:

$$A_3 = \frac{2}{V_x^2} (b^2C_{af} + c^2C_{ar}) \quad (17)$$

$$A_4 = 2(bC_{af} + cC_{ar}) \quad (18)$$

$$\frac{J_z}{V_x} \dot{a}_L + A_3 a_L - J_z \ddot{\beta} + A_4 \beta = 2bC_{af}\delta \quad (19)$$

The equation (13) can also have its derivatives with respect to time calculated:

$$\beta = \frac{2C_{af}\delta - A_1 a_L}{A_2} \rightarrow \dot{\beta} = \frac{2C_{af}\dot{\delta} - A_1 \dot{a}_L}{A_2} \rightarrow \ddot{\beta} = \frac{2C_{af}\ddot{\delta} - A_1 \ddot{a}_L}{A_2} \quad (20)$$

Applying the results above in (19), it is then possible to obtain the following:

$$\begin{aligned} \frac{J_z}{V_x} \dot{a}_L + A_3 a_L - J_z \left(\frac{2C_{af}\ddot{\delta} - A_1 \ddot{a}_L}{A_2} \right) + A_4 \left(\frac{2C_{af}\delta - A_1 a_L}{A_2} \right) &= 2bC_{af}\delta \\ \rightarrow \frac{A_1}{A_2} J_z \ddot{a}_L + \frac{J_z}{V_x} \dot{a}_L + \left(A_3 - \frac{A_1 A_4}{A_2} \right) a_L &= 2C_{af} \left(b - \frac{A_4}{A_2} \right) \delta + \frac{2C_{af} J_z}{A_2} \ddot{\delta} \end{aligned} \quad (21)$$

Taking into consideration that the steering wheel excitation studied is periodic in this case, it is possible to take its derivative with respect to time:

$$\ddot{\delta} = -\omega^2 \delta \quad (22)$$

Finally, with the help of additional auxiliary variables A_5 to A_8 , the following results are obtained:

$$A_5 = \frac{A_1}{A_2} J_z \quad (23)$$

$$A_6 = \frac{1}{V_x} J_z \quad (24)$$

$$A_7 = A_3 - \frac{A_1 A_4}{A_2} \quad (25)$$

$$A_8 = 2C_{af} \left(b - \frac{(A_4 + J_z \omega^2)}{A_2} \right) \quad (26)$$

$$A_5 \ddot{a}_L + A_6 \dot{a}_L + A_7 a_L = A_8 \delta \quad (27)$$

It is possible to notice that the equation (27) obtained for the lateral acceleration response of the vehicle to periodic steering wheel excitation is completely analogous to a simple mass-spring-damper linear system, with the difference that the term A_5 has units of kg ms^2 , A_6 has units of kg ms , A_7 has units of kg m and A_8 has units of Nm . Keeping these differences in mind, it is possible to obtain the analogue terms for natural frequency ω_n , damping coefficient ζ and frequency ratio r :

$$\omega_n = \sqrt{\frac{A_7}{A_5}} \quad (28)$$

$$\zeta = \frac{A_6}{2\sqrt{A_7 A_5}} \quad (29)$$

$$r = \frac{\omega}{\omega_n} \quad (30)$$

The definition of the frequency response of a dynamic system states that $G(\omega) = \frac{\text{output}(\omega)}{\text{input}(\omega)}$. In this sense, using the steer angle as input and the lateral acceleration at the CG of the vehicle as the output for the system being considered and the known solution for the mass-spring-damper system, it is possible to obtain:

$$\begin{aligned} G_s(\omega) &= \left| \frac{A_8}{A_7} \right| \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \\ &= \left| \frac{2C_{\alpha f} \left(b - \frac{(A_4 - J_z \omega^2)}{A_2} \right)}{A_7} \right| \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n} \right)^2 \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2}} \end{aligned} \quad (31)$$

The result of the equation (31) has units of $\frac{\text{m/s}^2}{\text{rad}}$ and corresponds to the behaviour of the lateral acceleration response to periodic excitation as shown in Figure 2.

From the equations obtained, the null gain frequency happens when the term $A_8 = 0$, as follows:

$$b - \frac{(A_4 + J_z \omega_{null}^2)}{A_2} = 0 \rightarrow \omega_{null} = \sqrt{\frac{bA_2 - A_4}{J_z}} \quad (32)$$

From the lateral acceleration bandwidth definition:

$$-3dB = 20 \log_{10} \left(\frac{G_s}{G_s|_{\omega=0}} \right) \quad (33)$$

After some algebraic manipulations and the definition of additional auxiliary variables A_9 to A_{11} , the frequency of the lateral acceleration bandwidth is then defined as:

$$A_9 = \left[\frac{10^{0.3} J_z^2}{(A_2 b - A_4)^2} - \frac{1}{\omega_n^4} \right] \quad (34)$$

$$A_{10} = \left[\frac{(2 - 4\zeta^2)}{\omega_n^2} - \frac{2 \cdot 10^{0.3} J_z}{(A_2 b - A_4)} \right] \quad (35)$$

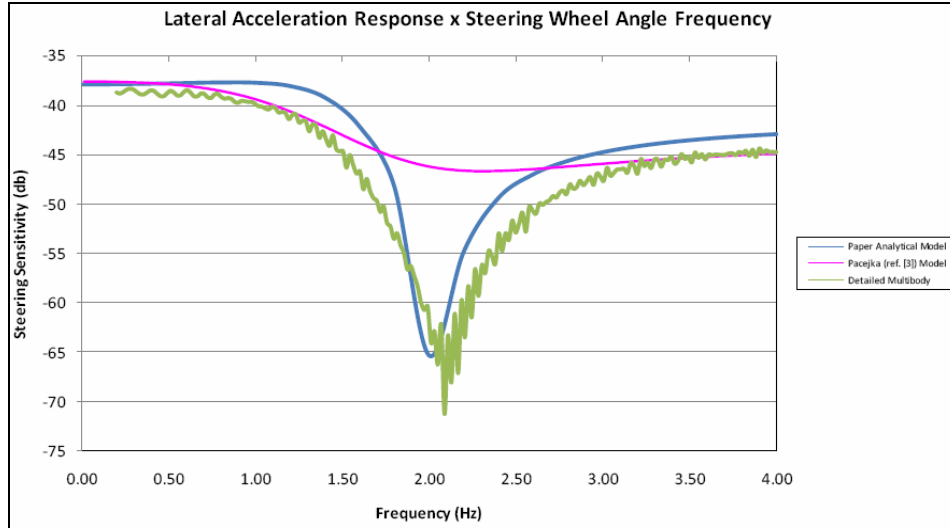
$$A_{11} = 10^{0.3} - 1 \quad (36)$$

$$\omega_{band}^2 = \frac{-A_{10} - \sqrt{A_{10}^2 - 4A_9 A_{11}}}{2A_9} \quad (37)$$

4 Analytical model results comparison

In order to quantify the accuracy of the proposed analytical model and the validity of the adopted simplifications, its results are compared against a detailed multibody model with 256 degrees of freedom like the model depicted in Figure 1 and the analytical solution proposed by Pacejka (2002) in the equation (1.78) from that work. This comparison has been performed for a passenger vehicle and the results are shown in Figure 6.

Figure 6 Lateral acceleration response \times frequency comparison – dB scale (see online version for colours)



Note: Proposed analytical model results in blue and detailed multibody model results in purple.

The results of the proposed analytical model are similar to the results obtained by Prado et al. (2001). All models analysed capture well the lateral acceleration bandwidth, which is the range where there is no significant variation from the steady state lateral acceleration response value. Ideally, this range would be as wide as possible, i.e., the vehicle has a response similar to the steady state response in a wide range of steering wheel frequency input. The results of the vehicle studied represent a regular passenger vehicle response, with the bandwidth between 1.0 Hz and 1.5 Hz.

However, compared to the analytical model proposed by Pacejka (2002), it is understood that the inclusion of the derivative of the sideslip angle β as shown in equation (4) in the proposed model leads to a better accuracy in capturing the null gain frequency effect. This null gain frequency effect may affect important aspects of the vehicle handling, as it represents a frequency value where the steering wheel input does not produce any lateral acceleration at all. The value around 2.0 Hz is also typical for passenger vehicles and the analytical model proposed here could capture this effect in a satisfactory way.

Finally, the Table 1 shows the comparison of the null gain frequency and lateral acceleration bandwidth metrics between the proposed analytical model and the detailed multibody model.

Table 1 Lateral acceleration frequency response metrics results comparison

	<i>Analytical model</i>	<i>Detailed multibody</i>	<i>Absolute difference</i>	<i>% difference</i>
Lateral acceleration bandwidth (Hz)	1.53	1.35	0.18	13.5%
Null gain frequency (Hz)	2.04	2.09	-0.05	-2.4%

It is important to mention that the results herein shown already take into consideration the following effects:

$$e1 = f(\text{tyre self-align torque}) \quad (38)$$

$$e2 = f(\text{vehicle's suspension and steering system compliances}) \quad (39)$$

$$e3 = f(\text{kinematic steering variation with vertical suspension travel}) \quad (40)$$

$$e4 = f(\text{lateral load transfer}) \quad (41)$$

All these effects already affect the steady state lateral acceleration response results (known as steering sensitivity in the literature) and their implementation in the analytical formulation proposed here can be done very straightforwardly through the substitution of the terms $C_{\alpha f}$ and $C_{\alpha r}$ by the equivalent terms $C'_{\alpha f}$ and $C'_{\alpha r}$ in the formulation previously described, which will include these effects in the final results, as follows:

$$C'_{\alpha f} = f(C_{\alpha f}, e1, e2, e3, e4) \quad (42)$$

$$C'_{\alpha r} = f(C_{\alpha r}, e1, e2, e3, e4) \quad (43)$$

These effects are described in more detail in the Appendix 1.

5 Conclusions

This paper has proposed a new analytical solution to the lateral acceleration response of a vehicle submitted to a periodic input at the steering wheel. The proposed method presents better accuracy than previously available analytical solutions (Pacejka, 2002) for the same problem.

The metrics related to the lateral acceleration response to periodic excitation have been presented and the results obtained through the proposed analytical model have been compared to a much more detailed multibody model. Small difference has been observed, showing that the proposed analytical model is capable of reproducing with good accuracy the more detailed multibody model. This allows its utilisation for development purposes, especially during early development phases for conceptual designs. It can be also used in numerical optimisation procedures, where the gains in computational running time are very interesting.

A future development proposal is the extension of the model herein proposed for inputs other than periodic steering variation, like step steering inputs and impulse steering inputs. For linear systems these responses can be simply derived through transformations of the frequency response functions. The analogy with a mass-spring-damper system used in the proposed approach might be of great interest for the design engineer, as this kind of simple mechanical system has easy dynamic understanding and classic analytical solution. Following the same steps, outputs other than lateral acceleration, like roll angle or yaw velocity, could also be calculated analytically for dynamic manoeuvres.

Finally, an experimental investigation to compare the results of a physical vehicle against the analytical model and the detailed multibody model is recommended in order to have a more complete understanding of correlation level for each modelling technique.

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Appendix 1*Definitions, acronyms and abbreviations*

X, Y, Z	absolute coordinate system (inertial)
x', y', z'	vehicle coordinate system (non-inertial)
CG	vehicle centre of gravity
O	centre of curve
R	curve radius
V_x	vehicle longitudinal velocity
V_y	vehicle lateral velocity
V_{CG}	vehicle CG total velocity
β	vehicle sideslip angle
Ω	turning angular velocity
H_{cg}	centre of gravity height to the ground
H_{reg}	roll centre at CG position
θ	vehicle roll angle
z_{roll}	roll moment arm
δ	front wheel steer angle
α_f / α_r	front/rear tyre slip angle
V_f / V_r	front/rear tyre velocity vector
L	wheelbase
b/c	distance between CG and front/rear axle

Appendix 2*Additional considerations for the lateral acceleration results**e1 Tyre self-align torque effect*

The tyre self-align torque comes from the fact that the resultant lateral force generated by the tyre is not coincident with the tyre geometric centre, but rather located in a different point in the longitudinal axis of the tyre. This distance is known as pneumatic trail t , and effectively changes the distances b and c between the lateral force application points and the CG of the vehicle as follows:

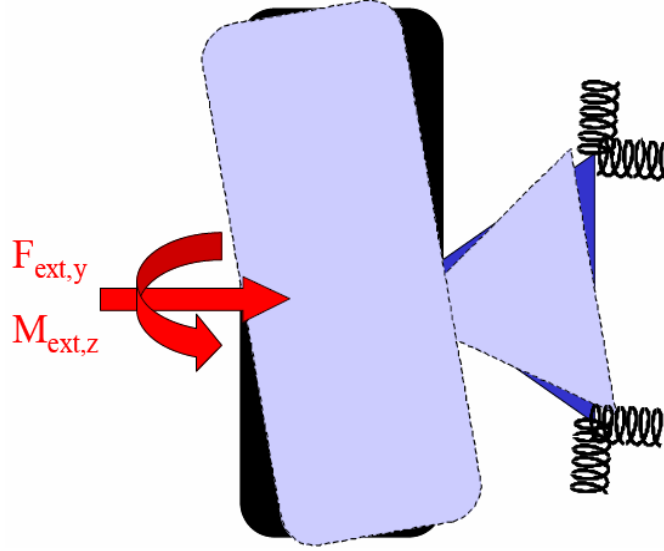
$$b' = b + t_f \quad (A1)$$

$$c' = c + t_r \quad (A2)$$

e2 Vehicle's suspension and steering system compliances

The forces and moments generated by the tyres cause deformations in the suspension and steering systems of the vehicle, as illustrated in Figure A1.

Figure A1 Effect of vehicle's suspension and steering system compliances (see online version for colours)



Assuming that there is a linear relationship between the tyre lateral force and align torque with the angle generated in the front/rear wheels due to the suspension and steering system compliance, the front/rear slip angles can be redefined as:

$$\alpha'_f = \alpha_f - F_{ext,yf} K_{fjf} - M_{ext,zf} K_{mzf} \quad (A3)$$

$$\alpha'_r = \alpha_r - F_{ext,yr} K_{fjr} - M_{ext,zr} K_{mzr} \quad (A4)$$

where the following definitions apply:

K_{fjf}, K_{fjr} front and rear wheel steer angle stiffness with respect to tyre lateral force

K_{mzf}, K_{mzr} front and rear wheel steer angle stiffness with respect to tyre align torque

$M_{ext,zf}, M_{ext,zr}$ front and rear tyre align torque.

Same as the lateral force, the front/rear tyre align torque is also assumed to be linear with respect to the tyre slip angle as follows:

$$M_{ext,zf} = 2C_{mz\alpha_f} \alpha_f \quad (A5)$$

$$M_{ext,zr} = 2C_{mz\alpha_r} \alpha_r \quad (A6)$$

It is possible to define then new auxiliary terms B_f and B_r :

$$B_f = 1 + 2C_{\alpha_f} K_{fjf} + 2C_{mz\alpha_f} K_{mzf} \quad (A7)$$

$$B_r = 1 + 2C_{\alpha_r} K_{fjr} + 2C_{mz\alpha_r} K_{mzr} \quad (A8)$$

And the slip angles adjusted by the suspension and steering system compliances can be then defined as:

$$B_f \alpha'_f = \alpha_f \quad (\text{A9})$$

$$B_r \alpha'_r = \alpha_r \quad (\text{A10})$$

e3 Kinematic steering variation with vertical suspension travel

The wheels also steer due to the vertical travel of the suspension, being this variation a function of the vehicle's specific suspension/steering geometry. This effect is shown in more detail by Milliken and Milliken (1995) in the chapter 19 and is also known in the literature as roll steer.

Considering that the vehicle is on a plane road, the vertical travel of the suspension is only a function of the vehicle roll angle θ , and the later can be considered linearly related to the lateral acceleration through the roll stiffness of the vehicle in the range of interest for this work (less than 0.4 g's of lateral acceleration). In this sense, following the same rationale previously described for the suspension and steering compliances, the kinematic steering variation with vertical suspension travel can be described through auxiliary terms $B_{f,rs}$ and $B_{r,rs}$, where the index **rs** refers to the roll steer effect. It is also interesting to mention that, in most cases, the front steered suspension is more sensitive to this effect than the rear suspension.

e4 Lateral load transfer

The lateral load transfer is a dynamic effect of the vehicle body under lateral acceleration, where there is a vertical (normal) load shift from the inner wheels to the outer wheels of the vehicle that is linearly proportional to the lateral acceleration that the vehicle is subject to and also the roll centre height of the front/rear suspensions (more details about roll centre height definition are shown by Milliken and Milliken (1995) in the chapter 17).

The effect in the equations herein developed is that the front and rear individual tyre cornering stiffness values C_{af} and C_{ar} are dependent on the tyre normal load. In this case, where the equations developed consider that the total cornering stiffness per axle is equal to two times the individual tyre cornering stiffness at static normal load, the correct consideration to take into account the lateral load transfer effect is to sum the inner and outer tyre cornering stiffness individually. This can be done by adopting the average of the inner and outer tyre values for the C_{af} and C_{ar} , as follows:

$$C_{af,llt} = \frac{C_{af,inner} + C_{af,outer}}{2} \quad (\text{A11})$$

$$C_{ar,llt} = \frac{C_{ar,inner} + C_{ar,outer}}{2} \quad (\text{A12})$$

In general, for low lateral acceleration values (that is the range of interest of this study), this effect is not as much important as the ones previously described in the Appendix section.

Summation of additional effects

The consideration of the effects previously described in the Appendix 1 for the tyre self-align torque, vehicle's suspension and steering system compliances, kinematic steering variation with vertical suspension travel and lateral load transfer can be implemented in the analytical solution proposed in the paper through the substitution of the terms C_{af} and C_{ar} by the equivalent terms C'_{af} and C'_{ar} in the formulation previously described, as follows:

$$C'_{af} = \frac{C_{af,lt}c}{B_f B_{f,rs} c'} \quad (A13)$$

$$C'_{ar} = \frac{C_{ar,lt}b}{B_r B_{r,rs} b'} \quad (A14)$$