NOTES AND PROBLEMS IN APPLIED GENERAL EQUILIBRIUM ECONOMICS

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Chapter 3

The Johansen Approach

3.1 Introduction

In this chapter we consider a class of general equilibrium models in which an equilibrium is a vector, V, of length n satisfying a system of equations

$$F(V) = 0, \qquad (3.1.1)$$

where F is a vector function of length m. We assume that F is differentiable and that the number of variables, n, exceeds the number of equations m. Via (3.1.1) consumer demands will be viewed as arising from budget-constrained utility maximization, zero pure profits will apply, and demands will equal supplies. Preferences and technologies are represented in (3.1.1) by differentiable utility and production functions.

Linearization will play a key role. We will be concerned with the approach pioneered by Johansen (1960). Because system (3.1.1) can be very large and involve a wide variety of nonlinear functional forms, from a computational point of view it might be quite intractable. Johansen's approach is to derive from (3.1.1) a system of linear equations in which the variables are changes, percentage changes or changes in the logarithms of the components of V.

Since system (3.1.1) contains more variables than equations we assign exogenously given values to (n–m) variables and solve for the remaining m, the endogenous variables. In applications of Johansen models, many different allocations of the variables between the exogenous and endogenous categories can be made. For example, if we are analyzing the effects of a change in the tariff on footwear, then this variable is exogenous. On the other hand, if we want to calculate the change in the tariff which would be required to ensure a given level of footwear employment, then the footwear tariff is an endogenous variable and footwear employment is exogenous.

For the purpose of introducing Johansen's computational approach, we can make some illustrative computations with a small system devoid of economic content. We will assume that system (3.1.1) consists of 2 equations and 3 variables and has the form

$$V_1^2 V_3 - 1 = 0 \qquad \text{and} \qquad V_1 + V_2 - 2 = 0. \qquad (3.1.2)$$

For our illustrative computations with (3.1.2), we will assume that the exogenous variable is V_3 and the endogenous variables are V_1 and V_2.

With this assignment of the variables to the exogenous and endogenous categories, we can express the endogenous variables as functions of the exogenous variable as follows:

$$V_1 = V_3^{-1/2} \qquad \text{and} \qquad V_2 = 2 - V_3^{-1/2} \tag{3.1.3}$$

where we assume (as is often the case in economic models) that only positive values for the variables are of interest. With a solution system such as (3.1.3), we have no difficulty in evaluating the effects on the endogenous variables of shifts in the exogenous variable. For example, assume that initially we have

$$V^I = (V_1^I, V_2^I, V_3^I) = (1,1,1), \tag{3.1.4}$$

a situation which satisfies (3.1.2). Then we want to evaluate the effects on V_1 and V_2 (employment and prices, say) of a shift in V_3 (the level of protection) from 1 to 1.1. By substituting into (3.1.3), we find that the new values for V_1 and V_2 are 0.9535 and 1.0465. We conclude that a 10 per cent increase in V_3 induces a 4.65 per cent reduction in V_1 and a 4.65 per cent increase in V_2.

Johansen-style computations make use of an initial solution, V^I, with results being reported usually as percentage deviations from this initial solution. The initial solution (i.e., the set of initial values for prices, quantities, tariffs, etc.) is known from the input-output data used in setting many of the parameters of the model (see Exercise 3.3). However, the computations for a Johansen model differ from the simple approach using (3.1.3) because the complexity and size of the system (3.1.1) normally rule out the possibility of deriving from it explicit solution equations. Instead, in the Johansen approach we solve a linearized version of (3.1.1).

To obtain the linearized version, we first derive from (3.1.1) a differential form

$$A(V)\,v = 0, \tag{3.1.5}$$

where $A(V)$ is an $m \times n$ matrix whose components are functions of V. The $n \times 1$ vector v is usually interpreted as showing percentage changes or changes in the logarithms of the variables V. However, in some models v is interpreted as the vector of changes in V. In the former case, $A(V)$ is chosen so that (3.1.5) can be used in evaluating elasticities of endogenous variables with respect to exogenous variables. In the latter case, (3.1.5) can be used in evaluations of derivatives. In either case, the linearized (and approximate) version of (3.1.1) used in a Johansen-style computation is generated by replacing the variable matrix $A(V)$ on the LHS of (3.1.5) by a fixed matrix, usually $A(V^I)$.

The derivation of (3.1.5) is by total differentiation of either (3.1.1) or a transformed version of it. The procedure can be illustrated in the context of (3.1.2). We totally differentiate the LHSs of (3.1.2) and set these total differentials to zero recognizing that if (3.1.2) is to continue to be satisfied after a disturbance in the exogenous variables, then the changes in the LHSs must be zero. Thus, we write

$$\begin{bmatrix} 2V_1V_3 & 0 & V_1^2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} dV_1 \\ dV_2 \\ dV_3 \end{bmatrix} = 0. \qquad (3.1.6)$$

This is a system of the form (3.1.5) where v is interpreted as the vector of changes in the variables V. To obtain a system where v is a vector of percentage changes, we transform (3.1.6) into[1]

$$\begin{bmatrix} 2 & 0 & 1 \\ V_1/2 & V_2/2 & 0 \end{bmatrix} \begin{bmatrix} 100(dV_1)/V_1 \\ 100(dV_2)/V_2 \\ 100(dV_3)/V_3 \end{bmatrix} = 0 \ . \qquad (3.1.7)$$

If we replace the (dV_i/V_i)s in (3.1.7) by $(d\ln V_i)$s then, on dividing all equations by 100, we obtain a system of the form (3.1.5) in which v is the vector of changes in the logarithms of V:

$$\begin{bmatrix} 2 & 0 & 1 \\ V_1/2 & V_2/2 & 0 \end{bmatrix} \begin{bmatrix} d\ln V_1 \\ d\ln V_2 \\ d\ln V_3 \end{bmatrix} = 0 \ . \qquad (3.1.8)^2$$

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1 The first equation in (3.1.7) is derived by dividing the first equation in (3.1.6) by $V_1^2V_3$. In the second equation in (3.1.7) we have divided through by 2. It is customary to use share coefficients in the linear-percentage-change system. Notice that $V_1/2$ and $V_2/2$ are the shares of V_1 and V_2 in the sum of V_1 and V_2.

2 More formally, we can derive (3.1.8) by first transforming (3.1.2) into

$$2\ln V_1 + \ln V_3 = \ln(1), \qquad \text{and} \qquad exp(\ln V_1) + exp(\ln V_2) = 2.$$

Then by total differentiation, we obtain

$$2d\ln V_1 + d\ln V_3 = 0, \qquad \text{and} \qquad V_1\, d\ln V_1 + V_2 d\ln V_2 = 0,$$

leading to (3.1.8). You will find later in this section that although the A(V) matrices in the percentage-change and log-change versions of (3.1.5) are identical, the two versions give different approximations for the effects on the endogenous variables of finite changes in the exogenous variables.

In a Johansen computation, a system of the form (3.1.5) effectively replaces (3.1.1) as the model. In computations of how far the endogenous variables will move from their initial values in response to given movements in the exogenous variables, A(V) is evaluated at $V = V^I$. Then (3.1.5) is rewritten as

$$A_\alpha(V^I)\, v_\alpha + A_\beta(V^I)\, v_\beta = 0, \tag{3.1.9}$$

where v_α is the $m \times 1$ subvector of endogenous components of v, v_β is the $(n - m) \times 1$ subvector of exogenous components and $A_\alpha(V^I)$ and $A_\beta(V^I)$ are appropriate submatrices of $A(V^I)$, i.e., $A_\alpha(V^I)$ is the $m \times m$ matrix formed by the columns of $A(V^I)$ corresponding to the endogenous variables and $A_\beta(V^I)$ is the $m \times (n - m)$ matrix formed by the columns corresponding to the exogenous variables. Finally, (3.1.9) is solved for v_α in terms of v_β giving[3]

$$v_\alpha = -A_\alpha^{-1}(V^I)\, A_\beta(V^I)\, v_\beta\,, \tag{3.1.10}$$

or more compactly

$$v_\alpha = B(V^I)\, v_\beta, \tag{3.1.11}$$

where $B(V^I)$ is defined by the right hand side of (3.1.10). If v is a vector of percentage changes or changes in logarithms, then the typical element, $B_{ij}(V^I)$, of $B(V^I)$ is the elasticity evaluated at V^I of the i^{th} endogenous variable with respect to the j^{th} exogenous variable. If v is a vector of changes, then $B_{ij}(V^I)$ is a derivative rather than an elasticity.

Computations (3.1.9) – (3.1.11) can be illustrated via systems (3.1.6) to (3.1.8). With $V = V^I = (1,1,1)$, (3.1.6) becomes

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} dV_1 \\ dV_2 \\ dV_3 \end{bmatrix} = 0. \tag{3.1.12}$$

On choosing variable 3 to be exogenous, we can rewrite (3.1.12) as

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} dV_1 \\ dV_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} dV_3 = 0. \tag{3.1.13}$$

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3 We assume that the relevant inverse exists. If this is not true, then the Johansen method will fail. However, if $A_\alpha(V^I)$ is singular, then it is likely that our classification of endogenous and exogenous variables is illegitimate. That is, it is unlikely that system (3.1.1) implies that V_α is a function of V_β in the region of V^I. In this case, any solution method should fail. See Dixon *et al.* (1982, section 35).

That is
$$\begin{bmatrix} dV_1 \\ dV_2 \end{bmatrix} = -\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dV_3 \ . \tag{3.1.14}$$

Hence
$$\begin{bmatrix} dV_1 \\ dV_2 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} dV_3 \ . \tag{3.1.15}$$

It is reassuring to note from (3.1.3) that when $V_3 = 1$,

$$\left[\frac{\partial V_1}{\partial V_3}\right] = -\frac{1}{2}(V_3)^{-3/2} = -0.5 \ , \quad \text{and} \quad \left[\frac{\partial V_2}{\partial V_3}\right] = \frac{1}{2}(V_3)^{-3/2} = 0.5.$$

We see that the 2×1 matrix of derivatives of the endogenous variables with respect to the exogenous variable evaluated at V^I is correctly revealed on the right hand side of (3.1.15). If we set $V = V^I$ in either (3.1.7) or (3.1.8), we can derive

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} 2 & 0 \\ 0.5 & 0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_3 \ ,$$

that is,
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} v_3 \ , \tag{3.1.16}$$

where (v_1, v_2, v_3) can be interpreted as either a vector of percentage changes or a vector of changes in logarithms. Again we can check our result by using (3.1.3) which gives

$$\eta_{1,3} = -\frac{1}{2}(V_3)^{-\frac{1}{2}}/V_1 \quad \text{and} \quad \eta_{2,3} = \frac{1}{2}(V_3)^{-\frac{1}{2}}/V_2 \ ,$$

where $\eta_{1,3}$ and $\eta_{2,3}$ are the elasticities of variables 1 and 2 with respect to variable 3. With $V = V^I$, we see that $\eta_{1,3} = -0.5$ and $\eta_{2,3} = 0.5$, confirming the result in (3.1.16).

Johansen's computational approach is an example of displacement analysis.[4] It allows us to evaluate derivatives or elasticities of endogenous variables with respect to exogenous variables without

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4 Many of you will be familiar with displacement analysis from derivations of the properties of demand elasticities in the utility maximizing model of consumer behaviour and the cost minimizing model of producer behaviour. See, for example, Dixon, Bowles and Kendrick (1980, Exercises 2.6 and 4.16).

having to obtain explicit forms for the solution equations [(3.1.3) in our example]. All that is required are some simple matrix operations. It should be emphasized, however, that these operations give us the values of the derivatives or elasticities only for the initial values, V^I , of the variables. When we move away from V^I, the derivatives or elasticities will change.

A little experimentation with (3.1.15) and (3.1.16) indicates that the Johansen approach is satisfactory for computing the effects on the endogenous variables of small changes in the exogenous variables. For example, by using (3.1.16) with the v_is interpreted as percentage changes we would say that a 10 per cent increase in V_3 would induce a 5 per cent reduction in V_1 and a 5 per cent increase in V_2. This is close to the answers (-4.65 and 4.65) which we found earlier by substituting into (3.1.3). Even greater accuracy is obtained in this particular example if we interpret the v_is as changes in logarithms. Then the exogenous shock is

$$d\,ln V_3 = ln(1.1) - ln(1) = 0.095310.$$

On applying this shock in (3.1.16) we obtain

$$d\,ln V_1 = -0.047655 \quad \text{and} \quad d\,ln V_2 = 0.047655, \qquad (3.1.17)$$

implying that V_1 and V_2 change by -4.65 and 4.88 per cent respectively.[5] However, when we make large changes in V_3, (3.1.16) may not give a satisfactory approximation to the effects on V_1 and V_2. Assume, for instance, that we increase V_3 by 100 per cent (i.e., from 1 to 2). Then the percentage-change version of (3.1.16) implies that V_1 will fall by 50 per cent to 0.5 and V_2 will increase by 50 per cent to 1.5. The correct values, derived from (3.1.3), are that V_1 will fall by 29.29 per cent to 0.7071 while V_2 will increase by 29.29 per cent to 1.2929. With the logarithmic version of (3.1.16), the shock is

$$d\,ln V_3 = ln(2) - ln(1) = 0.693147.$$

This produces $d\,ln V_1 = -0.346574$ and $d\,ln V_2 = 0.346574$. leading to the conclusion that the 100 per cent increase in V_3 reduces V_1 by 29.29 per cent and increases V_2 by 41.42 per cent. Although the logarithmic implementation of (3.1.16) has generated considerably greater accuracy

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5 The solution for the effect on V_1 of a change in V_3 is free from linearization error. This is because the solution function, (3.1.3), for V_1 can be written as $ln V_1 = -0.5 ln V_3$. Thus, no error is introduced by evaluating the change in $ln V_1$ as -0.5 times the change in $ln V_3$.

than the percentage change version, there is still an uncomfortably large error in the result for V_2.

When faced with a large change in the exogenous variables, one approach is to make a sequence of Johansen-style computations. For example, if we want to evaluate the effects of a 100 per cent increase in V_3, we can first use (3.1.16) to generate the effects of a 50 per cent increase. This would take us from the initial situation $(V = V^I)$ to one where $V = V^I + \Delta V_{50}$ with ΔV_{50} denoting our estimate of the change in V arising from the increase in V_3 from 1 to 1.5. Then we can reevaluate the elasticity matrix, B, at $V = V^I + \Delta V_{50}$ and use the reevaluated matrix in computing the effects of moving V_3 from 1.5 to 2. Where greater accuracy is required, we break the change in the exogenous variable into a larger number of smaller steps.

This extended or multi-step Johansen method is the subject of Exercises 3.7 and 3.8. It has been used by Dixon *et al.* (1982, sections 8 and 47). Their experience suggests that the original Johansen method is normally satisfactory. This finding is supported by Bovenberg and Keller (1981). It appears that in policy-oriented work, the changes in the exogenous variables are likely to be sufficiently small that no serious errors are introduced by treating the B matrix as a constant. In situations where it was necessary to allow the B matrix to move, Dixon *et al.* (again supported by Bovenberg and Keller) found that highly accurate solutions were obtained by applying their extended Johansen method with very few steps.

A final issue for this section concerns the interpretation of the changes, percentage changes or log changes in the linearized Johansen system. Johansen (1960) interpreted the variables of his linearized system of equations as growth rates. As described by Taylor (1975, p.100),

> "Basically, he proceeds by logarithmically differentiating the equations characterizing a Walrasian competitive equilibrium *with respect to time* in order to get a simultaneous system of equations which are linear in all growth rates."
> (Emphasis added)

Johansen was concerned mainly with forecasting; with making predictions about the development of the Norwegian economy over future periods. Nowadays, the more common use of Johansen models is in policy analysis in which the main concern is not the future state of the economy but how that state will be affected by, for example, the adoption of a proposal to increase protection against imports. Whereas the time-derivative interpretation of the variables is appropriate in forecasting, it is not appropriate for policy projections.

In forecasting, the initial solution (V^I) is interpreted as the actual state of the economy at time T where T is the current date or a recent date in history. Then the forecasts are made of growth rates in the exogenous variables relying on information from outside the model. For instance, demographic information might be used to forecast the growth of the labor force. Forecasts of movements in the foreign currency prices of imports and exports might be supplied by experts on particular commodity markets. For many exogenous variables, simple extrapolations from past trends might be used. Once a complete set of forecasts has been made for the exogenous variables for the period T to T + 10, say, the growth rates for the period in the endogenous variables can be forecast from the model.

Compared with forecasting, policy projections require little information on the vector of exogenous shocks, v_β. The appropriate values for the components of v_β are usually suggested in a straightforward way by the particular application at hand. For example, if we are interested in the effects of a 5 per cent increase in the real wage rate, then the percentage change in the real wage is set exogenously at 5 while all other components of v_β are set at zero. The model is then used to compute how different the endogenous variables would be from their levels in the vector V^I if the wage rate were 5 per cent higher; i.e., it is used to provide a comparison between two possible states of the economy at a given point of time, one with the real wage rate 5 per cent higher than the other.

3.2 Goals, Reading Guide and References

By the time you have finished with this chapter, we hope that you will have developed the basic skills required for constructing and using a Johansen-style general equilibrium model. In particular, we hope that you will

(1) be able to describe the four essential parts of the theoretical structure;

(2) understand the derivation of the linearized system from the nonlinear structural form;

(3) be familiar with the role of input-output tables in providing both the share coefficients for the linearized system and an initial solution for the nonlinear structural form;

(4) be able to distinguish the interpretation of the variables of the linearized system in forecasting applications from the interpretation which is appropriate for policy projections;

(5) be able to discuss the advantages and disadvantages of condensing the linearized system;

(6) be aware of various solution strategies for handling large sparse systems of linear equations;

(7) appreciate the flexibility inherent in being able to adopt different closures (choices of exogenous variables);

(8) be prepared to interpret solution matrices and to trace out the relationships between solution matrices computed for different closures;

(9) understand the source of the linearization errors occurring in Johansen computations;

(10) know how these linearization errors can be reduced to insignificance by a multi-step Johansen procedure supplemented by Richardson's extrapolation; and

(11) have a facility for deriving linearized demand and supply functions, suitable for use in a Johansen model, from a wide variety of utility maximizing, cost minimizing and revenue maximizing models.

Reading guide 3 and the problem set contain material which will help you to achieve these goals. The readings are referred to in abbreviated form. Full citations are in the reference list which also includes other references appearing in the chapter. The problem set is presented in three parts. Part A uses a small model to illustrate the basics of the Johansen approach. We suggest that you complete the problems in this part before doing any reading. Part B is concerned with linearization errors and their elimination. Part C will give you some practice in deriving linearized demand and supply functions. Exercises on more specialized aspects of Johansen models (e.g., the treatment of international trade, tariffs, taxes and investment) are included in Chapter 4.

Reading Guide to Chapter 3*

Begin

Johansen models employ the familiar technique of displacement analysis, i.e., a system of diferential equations is used to describe the displacement of equilibrium caused by the movements in exogenous variables. We suggest that you review a few applications of displacement analysis. Among the more famous are Meade's (1955) analysis of trade and welfare, Harberger's (1962) study of the effects of taxing capital, Frisch's (1959) and Houthakker's (1960) derivations of the restrictions on household demand functions flowing from the adoption of an additive utility function, and Jones' (1979) collected essays in the theory of international trade. Textbook treatments of displacement analysis include Intriligator (1971, particulary sections 7.4 and 8.3), Lancaster (1968, particulary chapters 4 and 8), and Dixon, Bowles and Kendrick (1980, particulary E2.5, E2.6 and E2.17).

Finished with the Stylized Johansen model of part A in the problem set and would now like to work through a more realistic illustrative model?

Yes

Read Dixon, Parmeter, Sutton and Vincent (DPSV) (1982, sections 3-7). Section 6 will be helpful if you want more examples of how the flexibility of a model is increased by making allowance for different closures. Also see Taylor *et al.* (1980, pp. 49-60) for a discussion of different closures in the context of an illustrative log change macro model.

No

Reading guide to Chapter 3 (continued)

No

Read Powell (1981) for a view of where Johansen models fit into the general field of economy-wide modeling. This paper also contains comments on why we build models, on forecasts versus policy projections and on many other modeling issues.

The classic work on Johansen models is of course Johansen (1960). You should at least glance through the whole book and read chapter 3.

Four modern examples of Johansen models are Taylor and Black (1974), Staelin (1976), Keller (1980) and DPSV (1982). Taylor and Black analyse the effects of changes in protection in Chile under a variety of production function specifications. Staelin's study is also a model of protection, this time applied to the Ivory Coast with an emphasis on noncompetitive pricing behavior. Keller presents a very carefully explained model of tax incidence in the Netherlands while DPSV describe a large multi-purpose model of Australia. We suggest that you review at least one of these models or any other operational Johansen model that interests you. Be sure to find out (i) how the model is used (forecasts or policy projections), (ii) how the share coefficients and other parameters are estimated, (iii) what closures are adopted, (iv) what is done about linearization errors and (v) what special features there are in theoretial structure and why they are included.

Reading guide to Chapter 3 (continued)

Interested in the linearization issue and want more information than is contained in part B of the problem set?

Yes

Look at Johansen (1960, pp. 217-222 in the 1974 edition only) which describes a dynamic correction method. A similar approach is adopted by Taylor *et al.* (1980, pp. 59-60). The approach described in part B of the problem set originates in DPSV (1982, particulary sections 8, 35 and 47).

No

Review goals listed in section 3.2. Exit if you have attained them.

Exit

* For full citations, see the reference list for this chapter.

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PROBLEM SET 3

A. *A STYLIZED JOHANSEN MODEL*

The implementation of a Johansen model typically includes the following steps:

(I) the development of a *theoretical structure* consisting of

  (i) equations representing household and other final demands for commodities,

  (ii) equations for intermediate and primary-factor inputs,

  (iii) pricing equations relating commodity prices to costs, and

  (iv) market clearing equations for primary factors and commodities;

(II) a *linearization* of the model equations to generate a system which is linear in percentage changes of the variables and in which most of the parameters are cost and sales shares;

(III) the use of *input-output* data to provide estimates for the relevant cost and sales shares; and

(IV) the development of *flexible computer programs* for condensing and manipulating linear systems.

In Exercises 3.1 – 3.6 we use a simple Johansen model to give you an overview of the four steps. We refer to this model as the Stylized Johansen model.

Exercise 3.1 *The theoretical structure for the Stylized Johansen model*

In this exercise, we ask you to derive the equations for a simple Johansen model. The model has two commodities, two primary factors and one final user (the household sector). We use the subscript 0 to refer to the final user. Subscripts 1 and 2 denote the two commodities and the two industries which produce them. Subscripts 3 and 4 refer to the primary factors labor and capital. We assume that:

(i) the household sector chooses its consumption levels of goods 1 and 2 (X_{10} and X_{20}) to maximize the Cobb-Douglas utility function

$$U = X_{10}^{\alpha_{10}} X_{20}^{\alpha_{20}} \tag{E3.1.1}$$

subject to the budget constraint

$$P_1X_{10} + P_2X_{20} = Y, \tag{E3.1.2}$$

where Y is the household expenditure level, and P_1 and P_2 are the prices of goods 1 and 2. α_{10} and α_{20} are positive parameters summing to one.

(ii) industry j, for j = 1 and 2, chooses its inputs X_{1j}, X_{2j}, X_{3j} and X_{4j} to minimize

$$C_j = \sum_{i=1}^{4} P_i X_{ij} \tag{E3.1.3}$$

subject to

$$X_j = A_j X_{1j}^{\alpha_{1j}} X_{2j}^{\alpha_{2j}} X_{3j}^{\alpha_{3j}} X_{4j}^{\alpha_{4j}} , \tag{E3.1.4}$$

where the X_{ij}s are the purchases of good 1, good 2, labor and capital by industry j; X_j is the output of good j by industry j; and A_j and the α_{ij}s are positive parameters with

$$\sum_{i=1}^{4} \alpha_{ij} = 1 .$$

Thus, we assume that whatever industry j's output level might be, the industry will minimize the costs of producing that output. In (E3.1.4) we assume that j's production technology is Cobb-Douglas.

(iii) our model accounts for all costs so that in each industry the value of output equals the value of the inputs. That is,

$$C_j = P_jX_j = \sum_{i=1}^{4} X_{ij}P_i \text{ , for } j = 1,2 . \tag{E3.1.5}$$

(iv) output levels for goods 1 and 2 (X_1 and X_2) and employment levels for labor and capital (X_3 and X_4) satisfy

$$\sum_{j=0}^{2} X_{ij} = X_i \quad , i = 1,2, \tag{E3.1.6}$$

and

$$\sum_{j=1}^{2} X_{ij} = X_i \quad , i = 3,4. \tag{E3.1.7}$$

Equation (E3.1.6) implies that demands equal supplies for goods 1 and 2. For primary factors, we simply assume that demands are satisfied, i.e., total employment of labor (X_3) is the sum of the demands for labor by the two industries. Similarly, the employment of capital (X_4) is the sum of the demands for capital by the two industries.

(v) the household budget (Y) equals factor income, that is

$$Y = P_3X_3 + P_4X_4 \quad . \tag{E3.1.8}$$

Now do the following:

(a) Show that the household demand functions are

$$X_{i0} = \alpha_{i0}Y/P_i, \qquad i = 1,2 \quad . \tag{E3.1.9}$$

(b) Prove that the production function (E3.1.4) exhibits constant returns to scale.

(c) Show that the input demand functions for industries 1 and 2 are given by

$$X_{ij} = (\alpha_{ij}Q_jX_j \prod_{t=1}^{4} P_t^{\alpha_{tj}}) / P_i \text{ ,} \quad i=1, \ldots, 4, \quad j=1,2 \tag{E3.1.10}$$

where

$$Q_j = \left[\prod_{t=1}^{4} (\alpha_{tj})^{-\alpha_{tj}} \right] / A_j \quad . \tag{E3.1.11}$$

(d) Show that α_{ij}, for i = 1, ..., 4 and j = 1,2, is the share of total costs in industry j represented by inputs of i.

(e) Derive from (E3.1.5) and (E3.1.10) the equations

$$P_j = Q_j \prod_{t=1}^{4} P_t^{\alpha_{tj}}, \qquad j = 1,2 . \qquad \text{(E3.1.12)}$$

What feature of the production functions (E3.1.4) is important in explaining why the zero-pure-profit conditions (E3.1.5) may be rewritten as relationships between prices with no quantity variables? That is, what allows us to eliminate the X_js and X_{ij}s in going from (E3.1.5) to (E3.1.12)?

(f) Show that once we have made assumptions (i) – (iv), then it is unnecessary to also include (v). In fact, (E3.1.8) is derivable from (E3.1.9) and (E3.1.5) – (E3.1.7). Thus, (E3.1.8) may be omitted from our description of the economy.

(g) Examine the system of equations (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6) and (E3.1.7). Assume that these equations are satisfied by

$\bar{X}_{i0}$, i=1,2; $\bar{X}_{ij}$, i=1, ..., 4, j=1,2; $\bar{X}_i$ and $\bar{P}_i$, i=1, ... 4; and $\bar{Y}$

Show that they continue to be satisfied when we modify this solution by multiplying all monetary variables (i.e., $\bar{P}_i$, i=1,...,4; and $\bar{Y}$) by any $\delta > 0$ while leaving all real variables unchanged.

(h) The system of equations (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6), (E3.1.7) and (E3.1.23)[6] is the structural form for our Stylized Johansen model. It corresponds to the system (3.1.1) in Section 3.1. How many variables do we have in our structural form? How many equations? Discuss possible closures. Would the model be adequately closed if we set P_3 and P_4 exogenously?

Answer to Exercise 3.1

(a) On putting the ratio of the marginal utilities of the two goods equal to the ratio of their prices, we find that

$$\alpha_{10} X_{10}^{\alpha_{10}-1} X_{20}^{\alpha_{20}} \; / \; \alpha_{20} X_{10}^{\alpha_{10}} X_{20}^{\alpha_{20}-1} = P_1 / P_2 .$$

This equation can be simplified and rearranged as

$$\alpha_{10} P_2 X_{20} = \alpha_{20} P_1 X_{10} .$$

6 (E3.1.23) is found in the answer to part (g) of this exercise.

Now we substitute P_1X_{10} out of the budget constraint (E3.1.2) to obtain

$$\left((\alpha_{10}/\alpha_{20}) + 1\right) P_2X_{20} = Y \ .$$

On recalling that $\alpha_{10} + \alpha_{20} = 1$, we establish (E3.1.9).

Notice that the αs are budget shares. Under a Cobb-Douglas utility function, the share of household expenditure going to each good is independent of commodity prices and the level of total expenditure.

(b) Imagine an initial situation in which the input levels are $\bar{X}_{ij}$, i = 1, ..., 4, giving an output of $\bar{X}_j$. Now assume that all input levels are multiplied by $\delta > 0$ leading to a new output level, $\bar{\bar{X}}_j$. (E3.1.4) implies that

$$\bar{X}_j = A_j \prod_{i=1}^{4} \bar{X}_{ij}^{\alpha_{ij}} \tag{E3.1.13}$$

and

$$\bar{\bar{X}}_j = A_j \prod_{i=1}^{4} \left(\delta \bar{X}_{ij}\right)^{\alpha_{ij}} \tag{E3.1.14}$$

Since $\sum_i \alpha_{ij} = 1$, we can rewrite (E3.1.14) as

$$\bar{\bar{X}}_j = \delta A_j \prod_{i=1}^{4} \bar{X}_{ij}^{\alpha_{ij}} \tag{E3.1.15}$$

Hence,

$$\bar{\bar{X}}_j = \delta \bar{X}_j \ . \tag{E3.1.16}$$

Equation (E3.1.16) shows that the new output level is δ times the old one. This establishes that (E3.1.4) exhibits constant returns to scale.

(c) The first-order conditions for industry j's cost minimization problem are

$$\alpha_{ij}(X_j/X_{ij}) = P_i/\lambda \quad \text{for } i=1, \ldots, 4 \tag{E3.1.17}$$

and

$$X_j = A_j \prod_{t=1}^{4} X_{tj}^{\alpha_{tj}} \ . \tag{E3.1.18}$$

where λ is the Lagrangian multiplier. To go from these five equations to the four input demand functions, we must eliminate λ. Our strategy is to obtain an expression for λ in terms of input prices and output. Then we substitute this expression back into (E3.1.17).

We start by rearranging (E3.1.17) as

$$X_{ij} = \lambda\alpha_{ij}X_j/P_i \ , \ i=1, \dots, 4. \qquad \text{(E3.1.19)}$$

Now we substitute from (E3.1.19) into (E3.1.18). On simplifying the resulting equation by taking into account that the αs sum to one, we find that

$$\lambda = Q_j \prod_{t=1}^{4} P_t^{\alpha_{tj}} \ , \qquad \text{(E3.1.20)}$$

where Q_j is defined in (E3.1.11). Finally we substitute from (E3.1.20) into (E3.1.17) to obtain (E3.1.10).

(d) We could work from the input demand functions, (E3.1.10). However, it is simpler to use (E3.1.17), from which we have

$$P_iX_{ij} \,/\, \textstyle\sum_t P_tX_{tj} = \lambda\alpha_{ij}X_j \,/\, \textstyle\sum_t \lambda\alpha_{tj}X_j \ . \qquad \text{(E3.1.21)}$$

Since the αs sum over the first subscript to one, the right hand side of (E3.1.21) simplifies to α_{ij}. Thus α_{ij} is the share of j's costs represented by inputs of i. Just as in part (a) we found that the Cobb-Douglas utility function implies constant budget shares, here we find that the Cobb-Douglas production function implies constant cost shares.

(e) By substituting from (E3.1.10) into (E3.1.5) we obtain

$$P_jX_j = \sum_{i=1}^{4} \alpha_{ij}Q_jX_j \prod_{t=1}^{4} P_t^{\alpha_{tj}} \ , \quad j=1,2. \qquad \text{(E3.1.22)}$$

Because the αs sum to one, (E3.1.22) simplifies to (E3.1.12).

The key to the elimination of the X's is the constancy of returns to scale in the production functions (E3.1.4). Equation (E3.1.5) says that the value of output equals the cost of inputs. Equivalently, we could say that average revenue per unit of output, P_j, equals the average cost per unit output. With a constant-returns-to-scale production function, the minimum average cost per unit of output can be calculated from the input prices. It is independent of the scale of output. Consequently, P_j is independent of the scale of output. The average cost curve is flat.

(f) On multiplying the i^{th} members of (E3.1.6) and (E3.1.7) through by P_i, and adding the resulting equations, we obtain

$$\sum_{i=1}^{2} P_iX_{i0} + \sum_{j=1}^{2}\sum_{i=1}^{4} P_iX_{ij} = \sum_{i=1}^{4} P_iX_i \ .$$

Next we substitute from (E3.1.9) and (E3.1.5). This gives

$$Y + \sum_{j=1}^{2} P_j X_j = \sum_{i=1}^{4} P_i X_i \, .$$

That is,

$$Y = P_3 X_3 + P_4 X_4 \quad .$$

This is an example of Walras' law. Once we have assumed that the total value of commodity outputs is equal to the total value of commodity demands (intermediate plus household) and that it is also equal to total costs (intermediate plus primary-factor), then we have implied that total household expenditure equals total payments to primary factors.

(g) When we use the modified solution to evaluate the left and right hand sides of (E3.1.9) we find that

$$\text{LHS} = \bar{X}_{i0} \quad \text{and} \quad \text{RHS} = \alpha_{i0}\delta\bar{Y} \,/\, \delta\bar{P}_i = \alpha_{i0}\bar{Y} \,/\, \bar{P}_i \quad .$$

Since the original solution satisfies (E3.1.9), we know that

$$\bar{X}_{i0} = \alpha_{i0}\bar{Y} \,/\, \bar{P}_i \quad .$$

Thus the modified solution satisfies (E3.1.9). We can establish similar results for (E3.1.10), (E3.1.12), (E3.1.6) and (E3.1.7).

We conclude that in the system (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6) and (E3.1.7), the absolute level of prices is indeterminate. It is often convenient to remove the indeterminacy by setting one of the prices at unity. We assume that

$$P_1 = 1 \, . \tag{E3.1.23}$$

Thus, good one becomes the numeraire or measuring stick. P_i will be the worth of good i in terms of units of good 1.

(h) Our structural form consists of 17 equations with 19 variables. To close the model we set values for two variables exogenously. One possible choice for the pair of exogenous variables is the primary factor employment levels, X_3 and X_4. This choice would be appropriate if, for example, we were interested in estimating the change in factor prices which would be required to allow a 10 per cent increase in the employment of labor over a period in which the capital stock in use was expected to increase by 5 per cent. Another possibility for the exogenous variables is P_3 and X_4. Here we might be interested in the effects of changes in wages, P_3, on the employment of labor, X_3, in the short run, i.e., a period sufficiently short for us to assume that the economy-wide capital stock, X_4, is determined independently of changes in wages.

A selection of exogenous variables which will not work is P_3 and P_4. This can be explained in at least two ways. First, look at the two-equation system (E3.1.12). This contains four variables P_1, P_2, P_3 and P_4. In part (g) we argued that P_1 can be set at unity and we added equation (E3.1.23) to our model. If we set P_3 and P_4 exogenously, we see that (E3.1.12) is a two equation system determining just one variable, P_2. Only by chance will there be a value for P_2 which is consistent with (E3.1.12), (E3.1.23) and exogenously given values for P_3 and P_4.

A second way to see that our model will not be closed adequately with P_3 and P_4 as exogenous variables is to think about what determines the size of the economy. If we did happen to have a solution for our model in which all of the Xs and Y were endogenous variables, then we would be able to generate further solutions simply by multiplying the Xs and Y by any $\delta > 0$. We would have nothing to tie down the absolute size of the economy. With P_3 and P_4 as our exogenous variables, we have over-determined the price side of our model and under-determined the real side.

Exercise 3.2 The percentage-change form of the Stylized Johansen model

Derive the percentage-change version of the structural form (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6), (E3.1.7) and (E3.1.23).

Answer to Exercise 3.2

In deriving the percentage-change form, we apply three rules:

The Product Rule, $R = \beta PQ \Rightarrow r = p + q$,

The Power Rule, $R = \beta P^{\alpha} \Rightarrow r = \alpha p$,

and *The Sum Rule,* $R = P+Q \Rightarrow r = pS_P+qS_Q$,

where r, p and q are percentage changes[7] in R, P and Q, α and β are parameters and S_P and S_Q are the shares of P and Q in P+Q, i.e.,

$$S_P = P \,/\, (P+Q) \quad \text{and} \quad S_Q = Q \,/\, (P+Q) \;.$$

7 They can, equally well, be interpreted as changes in logarithms.

Each of these rules is derived by totally differentiating the levels expression. In applying the rules, we must be careful not to divide by zero. Percentage-change or log-change forms are unsuitable for variables which have initial values of zero. To overcome this difficulty, it is sometimes convenient to work with transformed variables. For example, we might include in a model the power of a tariff (one plus the *ad valorem* rate) rather than the *ad valorem* rate. If the initial value of the *ad valorem* rate is zero, then the initial value of the power of the tariff is one. We will be able to calculate percentage changes or changes in the logarithm of the power of the tariff but not in the *ad valorem* rate.

In our Stylized Johansen model, we will assume that there are no variables whose initial values are zero. Therefore, we can apply our three rules directly to the structural form (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6), (E3.1.7) and (E3.1.23). We obtain

$$x_{i0} = y - p_i \ , \quad i = 1,2, \qquad \text{(E3.2.1)}$$

$$x_{ij} = x_j - (p_i - \sum_{t=1}^{4} \alpha_{tj} \, p_t) \ , \quad i = 1, \dots, 4, \ j = 1,2, \qquad \text{(E3.2.2)}$$

$$p_j = \sum_{t=1}^{4} \alpha_{tj} \, p_t \ , \quad j = 1,2, \qquad \text{(E3.2.3)}$$

$$\sum_{j=0}^{2} x_{ij} \, \beta_{ij} = x_i \ , \quad i = 1,2, \qquad \text{(E3.2.4)}$$

$$\sum_{j=1}^{2} x_{ij} \, \beta_{ij} = x_i \ , \quad i = 3,4, \qquad \text{(E3.2.5)}$$

and

$$p_1 = 0 \ , \qquad \text{(E3.2.6)}$$

where the lower case xs and ps can be interpreted either as percentage changes or log changes in the corresponding upper case variables, and

$$\beta_{ij} = X_{ij}/X_i \ , \quad i = 1, \dots, 4, \quad j = 0,1,2.$$

That is, the β_{ij}s are sales shares.

It is worth pausing to examine the system (E3.2.1) – (E3.2.6). Often the assumptions underlying a model are more clearly interpretable from the percentage-change form than from the original structural form. In the present model we see from equation (E3.2.1) that all household expenditure elasticities have the value 1, all own price elasticities are −1 and all cross price elasticities are zero. In anything beyond an illustrative model, a more realistic specification would be required, especially for the expenditure elasticities. Engel's law implies that expenditure elasticities for food are usually less than one, while those for clothing and consumer durables are usually greater than one — see Houthakker (1957). Consequently, for practical work we need more general descriptions of preferences than the Cobb-Douglas utility function (E3.1.1). Perhaps the most popular choice in applied general equilibrium modeling is the Klein-Rubin or Stone-Geary utility function leading to the linear expenditure system (see Dixon, Bowles and Kendrick, 1980, E2.3).

Equation (E3.2.2) says that in the absence of changes in relative prices, industry j will change the volumes of all its inputs by the same percentage as its output. This is a consequence of assuming constant returns to scale. On the other hand, if the percentage increase in the price of input i is greater than the percentage increase in a particular index of all input prices, then industry j will substitute away from input i. Its demand for input i will expand by less than its output. The weights used in the index of input prices are the cost shares, i.e., the αs. Finally in (E3.2.2), notice that the price-substitution term could have been written as $\sigma_j(p_i - \sum_t \alpha_{tj} p_t)$, where $\sigma_j = 1$. In other words our price-substitution term has an implied coefficient of one. This reflects the well-known property of Cobb-Douglas production functions that the elasticity of substitution between any pair of inputs is unity. Ideally, we should for applied work adopt more general production functions so that the coefficients on the substitution terms can vary according to the input substitution possibilities available in different industries. Production specifications are discussed further in Exercises 3.9 - 3.13.

Equation (E3.2.3) says that the percentage change in the price of good j is a weighted average of the percentage changes in input prices, the weights being cost shares. Equation (E3.2.4) says that the percentage change in the supply of commodity i is a weighted average of the percentage changes in various demands for i, the weights being sales shares. Similarly, (E3.2.5) equates the percentage change in the employment of factor i to a weighted average of the percentage changes in the industrial demands for i. The weights are the shares in the total employment of i contributed by each industry. The last equation, (E3.2.6), reflects our choice of good 1 as the numeraire.

Exercise 3.3 Input-output data and the initial solution

(a) Use the input-output data shown in Table E3.3.1 to evaluate the parameters of the structural form (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6), (E3.1.7) and (E3.1.23). That is, evaluate α_{i0} for i = 1,2; α_{ij} for i = 1, ..., 4 and j = 1,2; and Q_j for j = 1,2.

Hint: In evaluating the Q_js, assume that the quantity units underlying the flows in Table E3.3.1 are defined so that all prices are unity.

(b) Having evaluated the parameters of the structural form, we can check any suggested set of values for prices and quantities for consistency with our model. Check that the structural form equations are satisfied by the values in the input-output table, i.e., check that the model is solved by P_i = 1 for i = 1, ..., 4, X_{11} = 4, X_{21} = 2, X_{31} = 1, X_{41} = 1, X_1 = 8, X_{12} = 2, X_{22} = 6, X_{32} = 3, X_{42} = 1, X_2 = 12, X_{10} = 2, X_{20} = 4, Y = 6, X_3 = 4 and X_4 = 2.

Answer to Exercise 3.3

(a) From (E3.1.9) we know that the α_{i0}s are budget shares. For consistency with Table E3.3.1, they should be fixed at

$$\alpha_{10} = 2/6 = 0.\underline{3} \quad \text{and} \quad \alpha_{20} = 4/6 = 0.\underline{6}\ ,$$

where we use the notation $0.\underline{3}$ and $0.\underline{6}$ to denote 0.33... and 0.66... .

From (E3.1.10) we know that the α_{ij}s for i = 1, ..., 4 and j = 1,2 are cost shares. The values implied by Table E3.3.1 are

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \\ \alpha_{41} & \alpha_{42} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1\underline{6} \\ 0.25 & 0.5 \\ 0.125 & 0.25 \\ 0.125 & 0.08\underline{3} \end{bmatrix}$$

To evaluate the Q_js, we need to be able to tie down the A_js; see (E3.1.11). From (E3.1.4), it is clear that the values of the A_js depend on the units chosen for quantities of inputs and outputs. We adopt the convention that one unit of good or factor i is the amount which costs 1 dollar in the base period, i.e., the period to which our input-output data refer. Thus, without explicitly evaluating the A_js, we can conclude from (E3.1.12) that

$$Q_j = 1 \text{ for } j = 1,2.$$

Table E3.3.1

Input-Output Data (Flows in dollars)

		Industry 1	Industry 2	Households	Total Sales
Commodity	1	4	2	2	8
	2	2	6	4	12
Primary Factors	Labor 3	1	3		4
	Capital 4	1	1		2
Production		8	12	6	

(b) To check that the structural form equations are satisfied by the suggested values, we can substitute into left and right hand sides. For example, for i = 1, we have

LHS (E3.1.9) = 2 and RHS (E3.1.9) = $(0.\underline{3}) \times 6/1 = 2$.

Notice that our input-output data satisfy an important balancing condition. The total value of inputs for each industry equals the total value of sales. Where the share parameters[8] of a general equilibrium model are set to be consistent with a balanced input-output table, we can always use the table to deduce an initial solution to the structural form equations. The initial solution contains information which can be valuable in computing new solutions, especially if the exogenous shocks under consideration are not too large. It is a strength of the Johansen approach that it makes full use of the initial solution as a starting point for finding new solutions.

Exercise 3.4 Input-output data and the evaluation of $A(V^I)$

Complete the representation in Table E3.4.1 of the linearized system formed from (E3.2.1) – (E3.2.6) when the coefficients are evaluated using the input-output data from Table E3.3.1. That is, evaluate the $A(V^I)$ matrix.

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8 In the Stylized Johansen model, the share parameters of the nonlinear structural form (the αs) are simple cost and budget shares. When we move beyond Cobb-Douglas functions, then the share parameters (e.g., the δ's in the CES form, see Exercise 3.9) are less readily interpretable. It remains true, nevertheless, that when their values are set for consistency with a balanced input-output table, then the table reveals an initial solution to the structural form.

Table E3.4.1
The Transpose of the Matrix $A(V^I)$ for the Stylized Johansen Model: Incomplete*

	(E3.2.1)		(E3.2.2)								(E3.2.3)		(E3.2.4)		(E3.2.5)		(E3.2.6)
y	−1	−1															
x_{10}	1																
x_{20}		1															
x_{11}			1														
x_{21}																	
x_{31}																	
x_{41}																	
x_{12}																	
x_{22}																	
x_{32}																	
x_{42}																	
x_1			−1														
x_2																	
x_3																	
x_4																	
p_1	1		.5														
p_2		1	−.25														
p_3			−.125														
p_4			−.125														

* For typographical convenience we have listed the columns of $A(V^I)$ as rows.

Answer to Exercise 3.4

See Table E3.4.2.

Exercise 3.5 Condensing the Stylized Johansen model

In a detailed Johansen model, the dimensions, m and n, of A(V) may be very large. For example, in the *ORANI* model of the Australian economy both m and n are several million. Therefore, before we try to

Table E3.4.2

*Answer to Exercise 3.4: The Transpose of the Matrix $A(V^I)$ for the Stylized Johansen Model**

	(E3.2.1)		(E3.2.2)								(E3.2.3)		(E3.2.4)		(E3.2.5)		(E3.2.6)
y	−1	−1															
x_{10}	1												−.25				
x_{20}		1												$-.\underline{3}$			
x_{11}			1										−.5				
x_{21}				1										$-.1\underline{6}$			
x_{31}					1										−.25		
x_{41}						1										−.5	
x_{12}							1						−.25				
x_{22}								1						−.5			
x_{32}									1						−.75		
x_{42}										1						−.5	
x_1			−1	−1	−1	−1							1				
x_2							−1	−1	−1	−1				1			
x_3															1		
x_4																1	
p_1	1		.5	−.5	−.5	−.5	$.8\underline{3}$	$-.1\underline{6}$	$-.1\underline{6}$	$-.1\underline{6}$	.5	$-.1\underline{6}$					1
p_2		1	−.25	.75	−.25	−.25	−.5	.5	−.5	−.5	−.25	.5					
p_3			−.125	−.125	.875	−.125	−.25	−.25	.75	−.25	−.125	−.25					
p_4			−.125	−.125	−.125	.875	$-.08\underline{3}$	$-.08\underline{3}$	$-.08\underline{3}$	$.91\underline{6}$	−.125	$-.08\underline{3}$					

* For typographical convenience we have listed the columns of $A(V^I)$ as rows. Numbers of the form $.8\underline{3}$, $.1\underline{6}$, etc. are to be read as .8333..., .1666..., etc.

implement a solution of the form (3.1.10), it may be necessary to condense the linearized version of the model by eliminating some equations and variables. That is, starting from the $m \times n$ system

$$A(V)v = 0 \ ,$$

we derive a system of the form

$$A^*(V)v^* = 0$$

where A^* has the dimensions $(m-r) \times (n-r)$, v^* is a $(n-r)$ subvector of v and r is the number of eliminated variables.

(a) Condense the system (E3.2.1) - (E3.2.6) by eliminating household demands, x_{i0}, $i = 1,2$, and input demands, x_{ij}, $i = 1, \ldots, 4$, $j = 1,2$. That is, derive a seven equation system in the nine variables, x_i, p_i, $i = 1, \ldots, 4$ and y.

(b) Using the data from Table E3.3.1, compute the coefficient matrix $A^*(V^I)$ of the condensed system.

(c) In condensing a Johansen model, what criteria would you apply in selecting the variables to be eliminated?

Answer to Exercise 3.5

(a) We substitute the right hand sides of (E3.2.1) and (E3.2.2) into (E3.2.4) and (E3.2.5). The resulting 7 × 9 condensed system consists of (E3.2.3), (E3.2.6), plus

$$(y-p_i)\,\beta_{i0} + \sum_{j=1}^{2} [x_j - (p_i - \sum_{t=1}^{4} \alpha_{tj}\, p_t)]\, \beta_{ij} = x_i, \quad i = 1,2, \qquad \text{(E3.5.1)}$$

and

$$\sum_{j=1}^{2} [x_j - (p_i - \sum_{t=1}^{4} \alpha_{tj}\, p_t)]\, \beta_{ij} = x_i, \quad i = 3,4, \qquad \text{(E3.5.2)}$$

(b) See Table E3.5.1.

(c) First, we would avoid eliminating variables which we might want to set exogenously in some applications of the model. For example, we would not normally eliminate tax and tariff rates. In our Stylized Johansen model, we would not choose factor supplies or factor prices for elimination. Eliminated variables are necessarily endogenous.

Second, we would avoid eliminating key endogenous variables, those which are likely to be of interest when we are analyzing and presenting results. This criterion is not as important as the first. Eliminated variables can usually be recovered quite simply by back-solving. For example, if we used the condensed system (E3.2.3), (E3.2.6), (E3.5.1) and (E3.5.2) in computing solutions for our Stylized Johansen model, then by substituting values for x_i, p_i, $i = 1, \ldots, 4$ and y into (E3.2.1) and (E3.2.2) we could extend our solution to include the

Table E3.5.1
Answer to Exercise 3.5(b): The Matrix $A^*(V^I)$ for a Condensed Form of the Stylized Johansen Model

Equation Number	Variable								
	y	x_1	x_2	x_3	x_4	p_1	p_2	p_3	p_4
(E3.2.3)						.5	–.25	–.125	–.125
						$-.1\underline{6}$	.5	–.25	$-.08\underline{3}$
(E3.5.1)	–.25	.5	–.25			$.708\underline{3}$	–.25	–.125	$-.08\underline{3}$
	$-.\underline{3}$	$-.1\underline{6}$	.5			$-.1\underline{6}$	$.708\underline{3}$	$-.1458\underline{3}$	–.0625
(E3.5.2)		–.25	–.75	1		–.25	–.4375	.78125	–.09375
		–.5	–.5		1	–.3	–.375	–.1875	$.8958\underline{3}$
(E3.2.6)						1			

ten eliminated variables x_{i0}, $i = 1,2$ and x_{ij}, $i = 1, \ldots, 4$, $j = 1,2$. Nevertheless, back solving involves extra coding and computer time and it should be avoided if possible. Thus, we would include industry outputs and industry employment levels in the condensed system, whereas we might exclude intermediate input flows.

Third, we would try to keep the algebra simple. Ideal targets for elimination are variables which appear in no more than one or two equations and for which we have explicit expressions in terms of variables which are to be included in the condensed system. Commodity flows to households and input flows to industries often meet this criterion. For example, in the Stylized Johansen model, (E3.2.1) and (E3.2.2) provide simple explicit expressions for x_{i0}, $i = 1,2$ and x_{ij}, $i = 1, \ldots, 4$, $j = 1,2$ in terms of variables to be included in our condensed system. In addition, each of the x_{i0} and x_{ij} appear in only one other equation of the system (E3.2.1) – (E3.2.6), namely, in the relevant market-clearing equation.

How much condensing should we do? This depends on the programs we have available for solving linear systems. For example, with the *GEMPACK* software package[9], systems containing up to 1,000

9 See Codsi and Pearson (1988).

equations can be solved on commonly available personal computers. Hence, condensation is often unnecessary. Even for very large models still requiring condensation, *GEMPACK* removes the algebraic drudgery, users simply being required to specify which equations are to be used to eliminate which variables. These automated condensation procedures are less prone to error than use of pencil and paper.

Exercise 3.6 Two solution matrices for the Stylized Johansen model

In Exercise 3.3, we saw how the input-output data in Table E3.3.1 provide an initial solution, V^I, for our Stylized Johansen model. Then in Exercise 3.4, we evaluated the coefficients of the system (E3.2.1) - (E3.2.6) at V^I. This allows us to represent the model in the linearized form

$$A(V^I)v = 0, \qquad \text{(E3.6.1)}$$

where $A(V^I)$ is the 17×19 matrix whose transpose is shown in the body of Table E3.4.2 and v is the 19×1 vector of variables listed in the left margin of the table.

To solve the model we first choose two variables to be exogenous and we rearrange (E3.6.1) as in (3.1.9). Then, as in equations (3.1.10) and (3.1.11), we compute the 17×2 matrix $B(V^I)$. This is our solution matrix. The typical element shows the elasticity at V^I of the i^{th} endogenous variable with respect to the j^{th} exogenous variable.

In Table E3.6.1 we have given two solution matrices. The first was computed with the exogenous variables being x_3 and x_4 (employment of labor and capital). In this computation, the columns of $A_\alpha(V^I)$ are rows 1-13 and 16-19 of the transpose of $A(V^I)$ as displayed in Table E3.4.2 and $A_\beta(V^I)$'s columns are rows 14 and 15 of the same table. In the second computation the exogenous variables are p_3 and x_4 (the price of labor and the employment of capital). In going from the first to the second computation we switched column 14 out of the $A_\beta(V^I)$ matrix into the $A_\alpha(V^I)$ matrix and column 18 out of the $A_\alpha(V^I)$ matrix and into the $A_\beta(V^I)$ matrix. You might like to use the software and data on the companion diskettes described in Chapter 1 to carry out these two simulations for yourself and to check the results in Table E3.6.1.

In using a model, it is important to be able to explain the solution matrices in some detail. Convincing applications are possible only if we can isolate the particular aspects of the model which are responsible for particular results. In this exercise, your task is to explain various aspects of our two solution matrices for the Stylized Johansen model. Specifically, where $\eta_r(R,S)$ denotes the elasticity of endogenous variable R with respect to exogenous variable S in computation r (for example, $\eta_1(Y,X_3)$ is 0.6, $\eta_2(X_{10},P_3)$ is −1.5, etc.), account for the following relationships which are apparent in Table E3.6.1:

(a) $$\eta_r(P_1,V) = 0 \qquad \text{(E3.6.2)}$$
where V is any exogenous variable and r = 1,2.

(b) $$\eta_r(Y,V) = \eta_r(X_{10},V), \qquad \text{(E3.6.3)}$$
and
$$\eta_r(Y,V) = \eta_r(X_{20},V) + \eta_r(P_2,V) \qquad \text{(E3.6.4)}$$
where V is any exogenous variable and r = 1,2.

(c) $$\eta_1(P_2,X_3) < 0. \qquad \text{(E3.6.5)}$$

(d) $$\eta_1(V,X_3) + \eta_1(V,X_4) = 1 \qquad \text{(E3.6.6)}$$
where V is any endogenous quantity or income variable, and
$$\eta_1(V,X_3) + \eta_1(V,X_4) = 0 \qquad \text{(E3.6.7)}$$
where V is any endogenous price variable.

(e) $$\eta_2(V,X_4) = 1 \qquad \text{(E3.6.8)}$$
where V is any endogenous quantity or income variable, and
$$\eta_2(V,X_4) = 0 \qquad \text{(E3.6.9)}$$
where V is any endogenous price variable.

(f) $$\eta_2(V,P_3) = \eta_1(V,X_3)/\eta_1(P_3,X_3), \qquad \text{(E3.6.10)}$$
$$\eta_2(V,X_4) = \eta_1(V,X_4) - \eta_1(V,X_3)\,\eta_1(P_3,X_4)/\eta_1(P_3,X_3), \qquad \text{(E3.6.11)}$$
$$\eta_2(X_3,P_3) = 1/\eta_1(P_3,X_3), \qquad \text{(E3.6.12)}$$
and
$$\eta_2(X_3,X_4) = -\eta_1(P_3,X_4)/\eta_1(P_3,X_3), \qquad \text{(E3.6.13)}$$
where V is any variable which is endogenous in both computations 1 and 2. Can you see any practical application for relationships such as (E3.6.10) – (E3.6.13)?

(g) $$\eta_1(X_{31},X_3) = \eta_1(X_{32},X_3) = 1, \qquad \text{(E3.6.14)}$$
$$\eta_1(X_{41},X_3) = \eta_1(X_{42},X_3) = 0, \qquad \text{(E3.6.15)}$$
$$\eta_1(X_{31},X_4) = \eta_1(X_{32},X_4) = 0, \qquad \text{(E3.6.16)}$$
$$\eta_1(X_{41},X_4) = \eta_1(X_{42},X_4) = 1, \qquad \text{(E3.6.17)}$$
$$-\eta_1(P_3,X_3) + \eta_1(P_4,X_3) = 1, \qquad \text{(E3.6.18)}$$
and
$$\eta_1(P_3,X_4) - \eta_1(P_4,X_4) = 1. \qquad \text{(E3.6.19)}$$

Table E3.6.1

Solutions for the Stylized Johansen Model under Alternative Closures

			(1) Exogenous factor employment		(2) Exogenous wages and capital employment	
Variable Number			14	15	18	15
Elasticity of ↓		with respect to →	X_3 employment of labor	X_4 employment of capital	P_3 price of labor	X_4 employment of capital
1	Y	Household expenditure	0.6	0.4	-1.5	1
2	X_{10}	Household	0.6	0.4	-1.5	1
3	X_{20}	demands	0.7	0.3	-1.75	1
4	X_{11}	Intermediate	0.6	0.4	-1.5	1
5	X_{21}	and primary	0.7	0.3	-1.75	1
6	X_{31}	factor inputs to	1	0	-2.5	1
7	X_{41}	industry 1	0	1	0	1
8	X_{12}	Intermediate	0.6	0.4	-1.5	1
9	X_{22}	and primary	0.7	0.3	-1.75	1
10	X_{32}	factor inputs to	1	0	-2.5	1
11	X_{42}	industry 2	0	1	0	1
12	X_1	Commodity	0.6	0.4	-1.5	1
13	X_2	supplies	0.7	0.3	-1.75	1
14	X_3	Employment	N.A.	N.A.	-2.5	1
15	X_4	levels	N.A.	N.A.	N.A.	N.A.
16	P_1	Commodity	0	0	0	0
17	P_2	and factor	-0.1	0.1	0.25	0
18	P_3	prices	-0.4	0.4	N.A.	N.A.
19	P_4		0.6	-0.6	-1.5	0

N.A. (not applicable). The variable indicated in the row is exogenous.

Answer to Exercise 3.6

(a) Recall from (E3.2.6) that the price of good 1 is fixed in all computations.

(b) Equations (E3.6.3) and (E3.6.4) follow from the household demand equations (E3.2.1). For interpreting (E3.6.3), it is again necessary to recall that the price of good 1 is fixed.

(c) What we must explain is why an increase in the employment of labor, with the employment of capital held constant, reduces the price of good 2.

There are two avenues in the Stylized Johansen model for absorbing extra labor without changing the economy-wide employment of capital. First, there could be an increase in the labor/capital ratios of both[10] industries. This would require a reduction in the price of labor relative to that of capital leading to a reduction in the price of the labor intensive commodity relative to that of the capital intensive commodity. A glance at Table E3.3.1 is sufficient to convince us that good 2 is relatively labor intensive.

The second avenue is to increase the output of the labor intensive good (good 2) relative to that of the capital intensive good (good 1). Again this would require a reduction in P_2 relative to P_1. Otherwise, the change in the commodity composition of demands would not match the change in the composition of supply. Thus, with P_1 fixed, P_2 must fall if extra labor is to be absorbed through either avenue.

(d) Equations (E3.6.6) – (E3.6.7) imply that a one per cent increase in the employment of both scarce factors causes all real quantities and income to increase by one per cent with no changes in any prices. This reflects an absence of scale effects. In the Stylized Johansen model there are constant returns to scale in production and unitary income elasticities in consumption. Therefore, if we increase the employment of both labor and capital by one per cent, we can arrive at the new equilibrium without any changes in prices by

(i) increasing household income by one per cent causing

(ii) increases of one per cent in all household commodity demands which can be satisfied by

(iii) one per cent expansions in all commodity outputs which are made possible by

(iv) one per cent increases in all inputs (primary and intermediate).

(e) With the closure used in computation 2, capital is the only scarce factor. Equations (E3.6.8) and (E3.6.9) imply that if the wage rate is held constant, then a one per cent increase in the employment

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10 From (E3.2.2) we find that: $x_{31} - x_{41} = p_4 - p_3 = x_{32} - x_{42}$. Hence, the labor/capital ratios in the two industries cannot move in opposite directions.

of the scarce factor leads to a uniform one per cent expansion in the real side of the economy with no price changes. This result again reflects an absence of scale effects. Again we can arrive at the new equilibrium by a simple sequence. First we increase the employment of capital by one per cent in each industry without any changes in prices. Then we must increase all other inputs in both industries by one per cent – otherwise we would violate the cost minimizing input demand equations (E3.2.2). This means that we have increases of one per cent in the outputs of both commodities. Since the use of both commodities as intermediate inputs has increased by one per cent, we can be sure that there are one per cent increases in the quantities left over for household consumption. Finally, we note that the increase in factor employment has expanded household income by one per cent. Thus we have an equilibrium because the increase in the availability of commodities for household consumption is matched by the increase in household demand.

(f) $\eta_2(V,P_3)$ is the percentage change in variable V arising from a one per cent increase in the wage rate holding constant the employment of capital. Obviously we can compute $\eta_2(V,P_3)$ by adopting closure 2 and by setting $p_3 = 1$ and $x_4 = 0$. Alternatively we could adopt closure 1. Then the percentage change in variable V is given by

$$v = \eta_1(V,X_3)x_3 + \eta_1(V,X_4)x_4 \ . \tag{E3.6.20}$$

Also we have

$$p_3 = \eta_1(P_3,X_3)x_3 + \eta_1(P_3,X_4)x_4 \ . \tag{E3.6.21}$$

If we now want to compute the effect on V of a one per cent increase in wages with zero effect on the employment of capital, we can evaluate v in (E3.6.20) – (E3.6.21) with $p_3 = 1$ and $x_4 = 0$. This gives (E3.6.10).

To obtain (E3.6.11) we first note that $\eta_2(V,X_4)$ is the percentage change in variable V arising from a one per cent increase in the employment of capital, holding constant the wage rate. Thus, $\eta_2(V,X_4)$ may be found by computing v in (E3.6.20) – (E3.6.21) with $x_4 = 1$ and $p_3 = 0$. This gives

$$v = \eta_1(V,X_4) - \eta_1(V,X_3)\eta_1(P_3,X_4) \,/\, \eta_1(P_3,X_3),$$

establishing (E3.6.11).

Equation (E3.6.12) is derived by using (E3.6.21) to evaluate x_3 when $p_3 = 1$ and $x_4 = 0$. Finally (E3.6.13) follows if we evaluate x_3 in (E3.6.21) with $x_4 = 1$ and $p_3 = 0$.

Relationships such as (E3.6.10) – (E3.6.13) enable us to go from one closure to another without having to repeat the partitioning and solving steps described in (3.1.9) – (3.1.11). By applying these relationships to the results in Table E3.6.1 for closure 1, we can deduce any of the results for closure 2.

Computations similar to this are often useful in analysing simulation results. For example, imagine that we are trying to interpret a set of results on the effects of increases in tariffs computed under the assumption that the real wage rate adjusts to ensure that there is no change in aggregate employment. We may wish to see how the results are affected if we adopt the alternative assumption that it is employment which adjusts rather than the real wage rate. This requires a change of closure with aggregate employment becoming endogenous and the real wage rate becoming exogenous. By using relationships such as (E3.6.10) — (E3.6.13), results for key variables under the new closure can be computed conveniently with a pocket calculator.

(g) The first step in understanding (E3.6.14) – (E3.6.19) is to recognize that in the Stylized Johansen model the ratio of the value of output in industry 1 (Z_1) to that in industry 2 (Z_2) will never change. This would be obvious if there were no intermediate inputs. Then the values of outputs from industries 1 and 2 would equal the values of household demands for commodities 1 and 2. Under the Cobb-Douglas utility function, (E3.1.1), value shares in household expenditure are constant which would imply that value shares in total production would be constant also.

With intermediate inputs in the story, the constancy of Z_1/Z_2 depends on the Cobb-Douglas specification of the production functions as well as that of the utility function. The Cobb-Douglas production functions mean that in each industry the share of each input in the total value of output is constant. Thus, in the Stylized Johansen model implemented with the data in Table E3.3.1, we know that the value of commodity 1 used in the production of commodity 1 will always be $\frac{1}{2}Z_1$ and that value of commodity 1 used in the production of commodity 2 will always be $\frac{1}{6}Z_2$. Since the value of household consumption of commodity 1 will always be one third of total expenditure (Y), we can write:

$$Z_1 = \tfrac{1}{2}Z_1 + \tfrac{1}{6}Z_2 + \tfrac{1}{3}Y \quad . \tag{E3.6.22}$$

Similarly

$$Z_2 = \tfrac{1}{4}Z_1 + \tfrac{1}{2}Z_2 + \tfrac{2}{3}Y \quad , \tag{E3.6.23}$$

implying that

$$Z_1 = \tfrac{4}{3}Y \quad \text{and} \quad Z_2 = 2Y \quad , \tag{E3.6.24}$$

giving

$$Z_1/Z_2 = 2/3 \quad . \tag{E3.6.25}$$

Now that we have shown that Z_1/Z_2 is constant, it is also clear that X_{i1}/X_{i2} is constant for i = 1, ...,4. Remember that input value shares in Z_1 and Z_2 are constant and that input prices do not vary across industries. In particular, the employment of labor will always be allocated between the two industries in the base period proportions, i.e., 25 per cent to industry 1 and 75 per cent to industry 2. Similarly, the employment of capital will always be allocated 50 per cent to industry 1 and 50 per cent to industry 2. Therefore, if there is an x per cent increase in the aggregate employment of labor, there must be an x per cent increase in the employment of labor in each industry. If we put x equal to one, we have explained (E3.6.14) and, if we put it equal to zero we have explained (E3.6.16). Equations (E3.6.15) and (E3.6.17) follow in a similar way when we consider x per cent increases in the aggregate capital stock with x = 0 and x = 1. Finally, if there is an increase in the employment of labor of one per cent in each industry with no change in the employment of capital, then P_4/P_3 must increase by one per cent - otherwise there would be changes in the labor and capital shares in the values of industry output. Consequently we observe (E3.6.18). Similarly, if the employment of capital increases by one per cent in each industry with no change in the employment of labor, then P_3/P_4 must increase by one per cent. This leads to (E3.6.19).

B. ELIMINATING JOHANSEN'S LINEARIZATION ERRORS

Given a vector V which satisfies the structural equations (3.1.1), the Johansen method allows us to evaluate the derivatives or elasticities of the endogenous variables with respect to the exogenous variables. By totally differentiating the system (3.1.1) and applying the matrix operations described in (3.1.9) - (3.1.11) we obtain a matrix B(V) of either derivatives or elasticities at the point V. Johansen (1960) evaluated his B matrix at V^I, the vector of prices and quantities revealed by his base-period input-output data. He then calculated the effects on the endogenous variables (v_α) of changes in the exogenous variables (v_β) according to (3.1.11). The well-known weakness of this calculation is that it fails to allow for changes in the derivative or elasticity matrix, B(V), as V moves away from V^I.

The first step in overcoming this weakness is to recognize that we are dealing with a problem treated in detail in texts on numerical analysis. We have a system of the form

$$F(V_\beta, V_\alpha) = 0 \quad .$$

We assume that the system has a solution of the form

$$V_\alpha = G(V_\beta)$$

where $$F(V_\beta, G(V_\beta)) = 0$$

for all V_β in a neighborhood of an initial point, V^I_β. While we do not know the form of the G functions, we do know how to evaluate a matrix B(V) which has the property that

$$\nabla G(V_\beta) = B(V)$$

for all V satisfying the structural equations, where $\nabla G(V_\beta)$ is the matrix of partial derivatives of G and V_β is the exogenous subvector of V. Thus our problem is the standard one of numerical integration, i.e., given a starting point V^I and a formula for $\nabla G(V_\beta)$ evaluate

$$\Delta V_\alpha = G(V^F_\beta) - G(V^I_\beta)$$

where V^F_β and V^I_β are the final and initial values of the exogenous variables.

Having recognized the nature of our problem, we are free to solve it by using one of the numerous methods described in texts on numerical analysis.[11] These methods can be applied in our situation by multi-step Johansen procedures. In Exercises 3.7 and 3.8 we ask you to apply the Euler method where the shifts, $(V^F_\beta - V^I_\beta)$, in the exogenous variables are broken into n equal parts or possibly n equal percentage parts. Conceptually this is the simplest approach and it has, as was mentioned in Section 3.1, proved adequate in applications to the solution of general equilibrium models. Nevertheless, it would be possible in multi-step Johansen computations to adopt strategies which normally generate faster convergence to the true solution as we increase the number of steps, e.g. the strategy of Runge and Kutta, (see Cohen, 1973, Chapter 11).

Exercise 3.7 An introductory example of a multi-step Johansen computation

In this exercise we return to the system (3.1.2). Assume, as we did in Section 3.1, that V_3 is the exogenous variable and that initial values for the variables are given by (3.1.4).

(a) Use a two-step Johansen procedure to compute the effects on V_1 and V_2 of a 100 per cent increase in V_3. Base the calculations on (3.1.7), i.e., do the calculations using percentage changes in the variables. In the first step, calculate the effects on V_1 and V_2 of moving V_3 from 1 to 1.5. Then reevaluate the

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11 See for example, Cohen (1973), Dahlquist, Bjorck and Anderson (1974) and Conte and de Boor (1980).

elasticities of V_1 and V_2 with respect to V_3. In the second step, use the reevaluated elasticities in calculating the effects on V_1 and V_2 of moving V_3 from 1.5 to 2.

(b) Use a 4-step Johansen procedure to compute the effects on V_1 and V_2 of a 100 per cent increase in V_3. In the first step, increase V_3 from 1 to 1.25. In the second, increase V_3 from 1.25 to 1.50, etc. Continue to work with percentage changes rather than log changes.

(c) At this stage we have three Johansen-style estimates based on (3.1.7) of the values of V_1 and V_2 after a 100 per cent increase in V_3: the one-step estimate (0.5, 1.5) derived in Section 3.1 via equation (3.1.16) and the 2- and 4-step estimates obtained in parts (a) and (b) of this exercise. In Table E3.7.1, we have done some more arithmetic and added the 8-step estimate. Can you see a relationship between these four estimates? How could we extrapolate from the one- and two-step results to obtain improved estimates of the effects on V_1 and V_2 of a 100 per cent increase in V_3? Can you provide an extrapolation using all four sets of results?

Answer to Exercise 3.7

(a) In this example we have

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = B(V)\, v_3 \quad , \tag{E3.7.1}$$

where

$$B(V) = -\begin{bmatrix} 2 & 0 \\ V_1/2 & V_2/2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5V_1/V_2 \end{bmatrix} . \tag{E3.7.2}$$

We interpret the v_is as percentage changes.

In the first step of the two-step procedure we use

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{1,2} = B(V^I)50 = \begin{bmatrix} -25 \\ 25 \end{bmatrix}$$

as our estimate of the percentage effects on V_1 and V_2 of moving V_3 from 1 to 1.5. Thus, at the end of the first step, V has moved from (1,1,1) to

$$(V)_{1,2} = (0.75,\ 1.25,\ 1.5)$$

where we use the notation $(V)_{r,s}$ to denote the value of V at the end of the r^{th} step of an s-step procedure.

In the second step of the two-step procedure we use

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{2,2} = B\big((V)_{1,2}\big)\, 33\tfrac{1}{3} = \begin{bmatrix} -16.\underline{6} \\ 10 \end{bmatrix}$$

as our estimate of the percentage effects on V_1 and V_2 of moving V_3 from 1.5 to 2. Hence, our final estimate of V in the two-step procedure is

$$(V)_{2,2} = (0.625,\ 1.375,\ 2)\ . \qquad \text{(E3.7.3)}$$

On comparing (3.1.4) and (E3.7.3) we conclude that a 100 per cent increase in V_3 induces a 37.5 per cent reduction in V_1 and a 37.5 per cent increase in V_2.

(b) Our calculations give

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{1,4} = \begin{bmatrix} -12.5 \\ 12.5 \end{bmatrix} \quad \text{leading to } (V)_{1,4} = (0.875,\ 1.125,\ 1.25),$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{2,4} = \begin{bmatrix} -10 \\ 7.\underline{7} \end{bmatrix} \quad \text{leading to } (V)_{2,4} = (0.7875,\ 1.2125,\ 1.5),$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{3,4} = \begin{bmatrix} -8.\underline{3} \\ 5.4124 \end{bmatrix} \quad \text{leading to } (V)_{3,4} = (0.7219,\ 1.2781,\ 1.75)\ ,$$

and finally

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{4,4} = \begin{bmatrix} -7.1429 \\ 4.0342 \end{bmatrix} \quad \text{leading to } (V)_{4,4} = (0.6703,\ 1.3297,\ 2).$$

We conclude from the four-step procedure that a 100 per cent increase in V_3 induces a 32.97 per cent reduction in V_1 and a 32.97 per cent increase in V_2.

(c) Let us denote the result for variable i from a procedure with step size h by $V_i(h)$. (For example, in Table E3.7.1, $V_2(\frac{1}{8}) = 1.3103$). We make two assumptions. First that

$$\lim_{h \to 0} V_i(h) = V_i^T \qquad \text{(E3.7.4)}$$

where V_i^T, i = 1,2, is the true value for variable i after we increase V_3 to 2. V_1^T and V_2^T can be derived from (3.1.3) and are shown in the last row of Table E3.7.1 as 0.7071 and 1.2929. Our second assumption is that $V_i(h)$ can be expressed as

$$V_i(h) = \sum_{r=0}^{\infty} a_{ir} h^r\ ,\ i = 1,2, \qquad \text{(E3.7.5)}$$

over the relevant range for h (in our example [0,1]).

Table E3.7.1

Solutions for V_1 and V_2 in the System (3.1.2) when V_3 is moved from 1 to 2: Calculations based on (3.1.7) [(a)]

Endogenous Variables	V_1	V_2
Initial Values	1	1
Estimated values after an increase in V_3 from 1 to 2		
1 - step computation	0.5	1.5
2 - step computation	0.625	1.375
4 - step computation	0.6703	1.3297
8 - step computation	0.6897	1.3103
1,2 step extrapolation [(b)]	0.75	1.25
1,2,4 step extrapolation [(c)]	0.7041	1.2959
1,2,4,8 step extrapolation [(d)]	0.7073	1.2927
Truth [(e)]	0.7071	1.2929

(a) The calculations were done in percentage changes with the change in V_3 divided into equal parts. For example, in the first step of the 2-step calculation, we set $100(dV_3)/V_3 = 50$ thus moving V_3 from 1 to 1.5. In the second step we set $100(dV_3)/V_3 = 33.\underline{3}$, moving V_3 from 1.5 to 2.

(b) Computed according to (E3.7.9).

(c) Computed according to (E3.7.14).

(d) Computed according to (E3.7.16).

(e) Computed using (3.1.3).

Assumption (E3.7.4) says that by making the step size sufficiently small, i.e., by taking a sufficient number of steps, we can get arbitrarily close to the true answer. In other words, our n-step procedure converges to the true solution as n becomes large. If you want to read about convergence conditions, look under Euler's method in an intermediate text on numerical analysis, e.g., Young and Gregory (1972, pp. 441-449) and Conte and de Boor (1980, pp. 359-362). Convergence conditions are studied in detail in the specific context of a Johansen model in Dixon *et al.* (1982, section 35).

Assumption (E3.7.5) relies on the idea that continuous functions can be approximated arbitrarily closely by polynomials of sufficiently high degree – see Young and Gregory (1972, p. 308).[12] Notice that (E3.7.4) and (E3.7.5) together imply that

$$V_i^T = a_{i0} \quad , \ i = 1,2 \ . \qquad \text{(E3.7.6)}$$

Now suppose that $V_i(h)$ can be approximated by

$$V_i(h) = a_{i0} + a_{i1}h \quad , \ i = 1,2 \ . \qquad \text{(E3.7.7)}$$

In (E3.7.7) we are assuming that the higher order terms in (E3.7.5) can be ignored in the relevant range for h. If (E3.7.7) were valid, then we would have

$$V_i(h/2) - V_i(h) = -(a_{i1}/2)h \quad , \ i = 1,2 \ . \qquad \text{(E3.7.8)}$$

In particular, we would have

$$V_i(1/2) - V_i(1) = -(a_{i1}/2) \ ,$$

$$V_i(1/4) - V_i(1/2) = -(a_{i1}/2)\,(1/2)$$

and

$$V_i(1/8) - V_i(1/4) = -(a_{i1}/2)\,(1/4) \quad , \ i = 1,2 \ .$$

Hence we would find that the gaps between the answers from the one- and two-step procedures would be twice the gaps between the answers from the two- and four-step procedures. Similarly, the two/four gaps would be twice the size of the four/eight gaps. On looking at Table E3.7.1 we see that these relationships are approximately satisfied. For example, the results for V_1 give

$$V_1(1/2) - V_2(1) = 0.125 \approx 0.0906 = 2(V_1(1/4) - V_1(1/2)) \ ,$$

and

$$V_1(1/4) - V_1(1/2) = 0.0453 \approx 0.0388 = 2(V_1(1/8) - V_1(1/4)) \ .$$

The importance of approximations such as (E3.7.7) is that they often allow us to achieve adequate accuracy with multiple-step Johansen computations even though our computer budget may be sufficient for only a small number of steps. Assume, for example, that we are able to make only a one-step computation and a two-step computation. In terms of our example, we are able to evaluate $V_i(1)$ and $V_i(1/2)$ for i = 1,2. Then (E3.7.7) suggests that we should estimate V_i^T by solving for a_{i0} in the equations

$$V_i(1) = a_{i0} + a_{i1},$$

$$V_i(1/2) = a_{i0} + a_{i1}/2 \ .$$

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12 It might be objected that h takes only the values 1, $\frac{1}{2}$, $\frac{1}{4}$, etc. and is not a continuous variable. To overcome this problem, we can imagine that if h is 0.4, for example, then our procedure is to increase V_3 from 1 to 1.4, then from 1.4 to 1.8 and finally from 1.8 to 2. If h = 0.7, we move V_3 from 1 to 1.7 and then from 1.7 to 2, etc.

That is, we should estimate V_i^T by extrapolation from our one- and two-step solutions according to

$$V_i^T = 2V_i(1/2) - V_i(1), \quad i = 1,2. \tag{E3.7.9}$$

The results of applying (E3.7.9) are shown in Table E3.7.1 in the row labelled *1,2 step extrapolation.*

If our computer budget is a little less limited so that we can afford to make one-, two- and four-step computations, then we can replace (E3.7.9) by a more sophisticated extrapolation equation. First, we replace (E3.7.7) by the improved approximation

$$V_i(h) = a_{i0} + a_{i1}h + a_{i2}h^2 \ . \tag{E3.7.10}$$

Then, assuming that we have computed $V_i(h)$, $V_i(h/2)$ and $V_i(h/4)$, we solve for a_{i0} in the system of equations

$$V_i(h) = a_{i0} + a_{i1}h + a_{i2}h^2 \ , \tag{E3.7.11}$$

$$V_i(h/2) = a_{i0} + (a_{i1}/2)h + (a_{i2}/4)h^2 \ , \tag{E3.7.12}$$

$$V_i(h/4) = a_{i0} + (a_{i1}/4)h + (a_{i2}/16)h^2 \ , \tag{E3.7.13}$$

The solution for a_{i0} can be obtained by first multiplying (E3.7.11) by –1, (E3.7.12) by 6, (E3.7.13) by –8 and then adding the resulting equations. This gives

$$-V_i(h) + 6V_i(h/2) - 8V_i(h/4) = -3a_{i0} \ ,$$

leading to the extrapolation equation

$$V_i^T = (8/3)\ V_i(h/4) - 2V_i(h/2) + (1/3)V_i(h) \ . \tag{E3.7.14}$$

Application of (E3.7.14) in our example with $h = 1$ gives the results shown in Table E3.7.1 in the row labelled *1,2,4 step extrapolation.*

When $V_i(h)$, $V_i(h/2)$, $V_i(h/4)$ and $V_i(h/8)$ are available, we can improve the approximation (E3.7.7) to

$$V_i(h) = a_{i0} + (a_{i1})h + a_{i2}h^2 + a_{i3}h^3 \ . \tag{E3.7.15}$$

Then following a strategy similar to that which lead to (E3.7.9) and (E3.7.14) we can derive the extrapolation equation

$$V_i^T = (64/21)V_i(h/8) - (56/21)V_i(h/4) + (14/21)V_i(h/2) - (1/21)V_i(h). \tag{E3.7.16}$$

Application of (E3.7.16) in Table E3.7.1 gives the results in the row labelled *1,2,4,8 step extrapolation.*

Readers who are familiar with the numerical-methods literature will recognize equations (E3.7.9), (E3.7.14) and (E3.7.16) as examples of Richardson's extrapolation.[13] Extrapolation techniques can usefully supplement any computational procedure where the aim is to evaluate F(h) in the limit as h approaches zero by computing a sequence $F(h_1)$, $F(h_2)$,... for $h_1 > h_2 > ... > 0$. Dahlquist, Bjorck and Anderson (1974, p. 270), in referring to an extrapolation procedure, comment that: "This process is, in many numerical problems — especially the treatment of integral and differential equations — the simplest way to get results which have negligible truncation error".

Exercise 3.8 A multi-step computation for the Stylized Johansen model

Figure E3.8.1 is a flow diagram for a multi-step solution of a Johansen model. To start the computations (box 1), we must read in the input-output data (Table E3.3.1 for our Stylized model). Normally, we would also read in various substitution parameters. In the Stylized model, this is not necessary. Under the Cobb-Douglas specifications in this model, all the substitution elasticities are unity, and need not appear explicitly in our computations. Other data which can be supplied at the initial stage of the computations are the closure (i.e., the choice of exogenous variables), the shocks (i.e., the changes in the exogenous variables) and the number of steps to be used (denoted by s). Finally, we set a counter, r, which will keep track of how many steps have been completed.

The arithmetic starts in box 2 with an evaluation of either an A matrix or a condensed version of one. Condensing is not necessary in the Stylized model. We will work with the system (E3.2.1) – (E3.2.6). With our counter, r, at zero, the A matrix is evaluated using the initial input-output data. We denote this initial A matrix by $A((V)_{0,s})$ where $(V)_{r,s}$ is the vector of values attained by the variables at the end of r steps of an s-step procedure. $(V)_{0,s}$, which has previously been denoted as V^I, reflects the prices and quantities implied by the initial input-output data. For our Stylized model, $A((V)_{0,s})$ was derived in Exercise 3.4 and is displayed in Table E3.4.2.

On reaching box 3 with r = 0, we compute the shocks to be made to the exogenous variables in the first step of the computation, i.e., we evaluate the vector $(v_\beta)_{1,s}$. Many sensible schemes are available for dividing the total change in each exogenous variable into s parts. For example, in Exercise 3.7(a) where s was 2, we broke the total change (from 1 to 2) in the exogenous variable (which was V_3) into a

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13 See especially Dahlquist, Bjorck and Anderson (1974, pp. 269-273).

pair of equal parts. Our first step was to compute the effects of moving V_3 from 1 to 1.5. In the second step we moved V_3 from 1.5 to 2. Because we interpreted the v_is as percentage changes, the total change in the V_3 was implemented as $(v_3)_{1,2} = 50$ followed by $(v_3)_{2,2} = 33.\underline{3}$.

Alternatively we could have broken the changes in the exogenous variable into equal percentage parts i.e.,

$$(v_3)_{r,2} = (\sqrt{2} - 1)100 = 41.4213 \qquad \text{for } r = 1,2. \qquad \text{(E3.8.1)}$$

Another possibility was to interpret the v_is as log changes. and to break the change in V_3 into equal logarithmic parts, i.e.,

$$(v_3)_{r,2} = \tfrac{1}{2}\,[ln(2) - ln(1)] = 0.34657 \qquad \text{for } r = 1,2\ . \qquad \text{(E3.8.2)}$$

We suspect that the choice between schemes such as equal changes and equal percentage or log changes is not often an important one.

Box 4 of Figure E3.8.1 is where the shifts in the endogenous variables at each step are computed. First, the A matrix is partitioned into A_α consisting of the columns corresponding to the endogenous variables, and A_β consisting of the columns corresponding to the exogenous variables. Then the system of equations

$$A_\alpha\big((V)_{r,s}\big)(v_\alpha)_{r+1,s} + A_\beta\big((V)_{r,s}\big)(v_\beta)_{r+1,s} = 0 \qquad \text{(E3.8.3)}$$

is solved for $(v_\alpha)_{r+1,s}$. This can be done by computing $B\big((V)_{r,s}\big)$ which is given by

$$B\big((V)_{r,s}\big) = -\left[\,A_\alpha\big((V)_{r,s}\big)\,\right]^{-1} A_\beta\big((V)_{r,s}\big)\ ,$$

and then post multiplying by $(v_\beta)_{r+1,s}$. In evaluating B matrices, computational costs can be kept low by avoiding the inversion of A_α. If B is a matrix of elasticities, the j^{th} column, $(B._j)$, can be computed by considering the effects on the endogenous variables of a one per cent increase in the j^{th} exogenous variable holding constant all other exogenous variables. If B is a matrix of derivatives, then we can consider the effects of a unit increase in the j^{th} exogenous variable. Thus, in either case, we can compute $B._j$ by applying efficient methods[14] to the solution of the system

$$A_\alpha B._j = -(A_\beta)._j \quad , \qquad \text{(E3.8.4)}$$

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14 In the context of Johansen models, these include Jacobi, Gauss-Seidel and other sparse matrix methods (see, for example, Tewarson, 1973) which take advantage of the fact that usually only a small fraction (less than 10 per cent) of the components of A_α are non-zero.

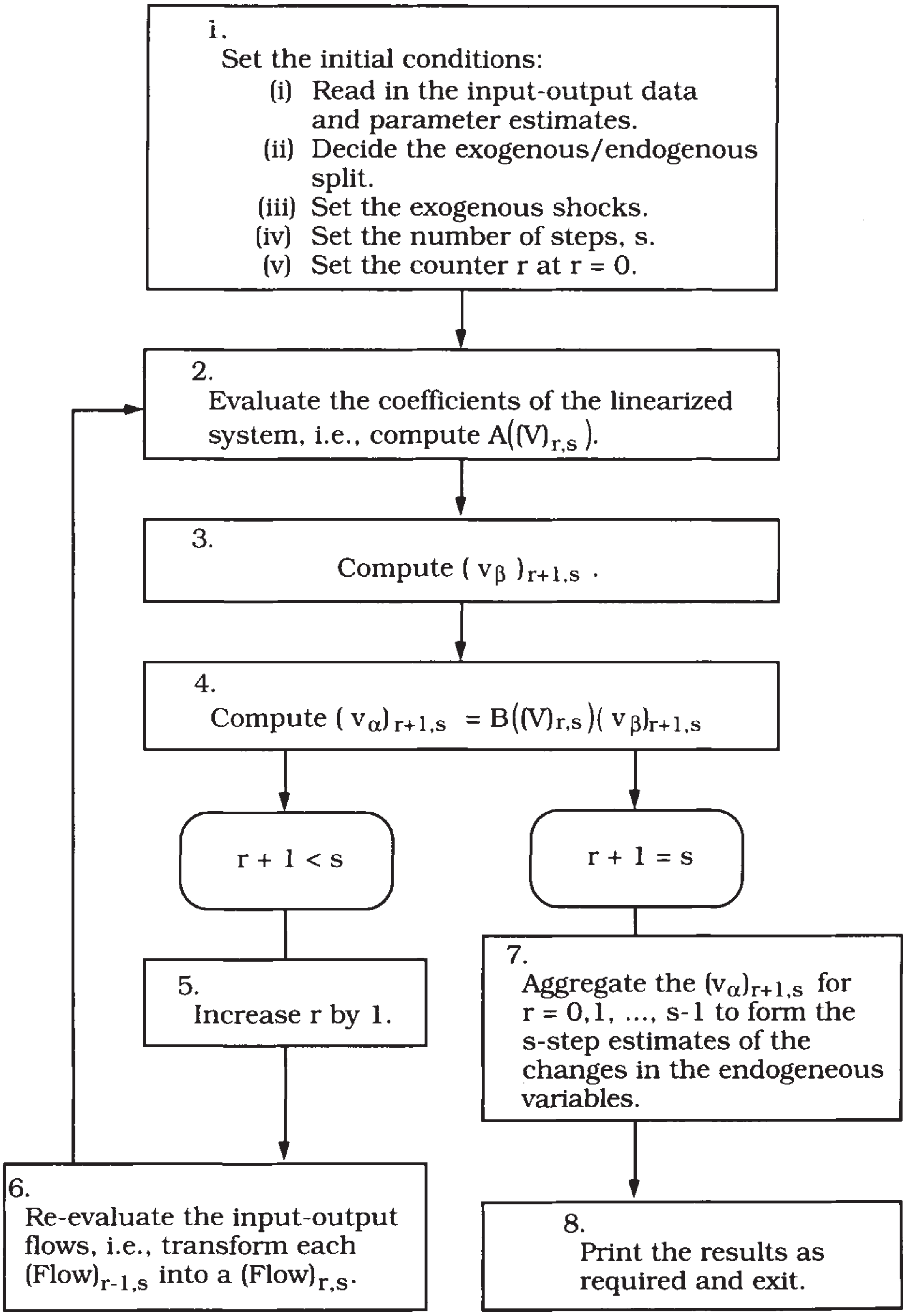

Figure E3.8.1 Flow diagram for a multi-step solution of a Johansen model where $(A_\beta)_{\cdot j}$ is the j^{th} column of A_β. In our Stylized model where there are only 2 exogenous variables, B matrices can be evaluated by solving just two systems of linear equations of the form (E3.8.4).[15] $B((V)_{0,s})$ matrices for two alternative closures of the Stylized model are displayed in Table E3.6.1.

With the completion of the work in box 4, we have come to the end of the $(r+1)^{th}$ step of our computation. Assuming that r+1 is less than s, we move to box 5. There we increase r by 1 and we commence the next step.

Our first task (box 6) in the new step is to update the input-output data taking account of changes in prices and quantities occurring in the previous step. For example, if we are just commencing the second step (r=1), then we will be concerned with how each input-output flow has been changed from its initial value, $(Flow)_{0,s}$, by the changes in prices and quantities in the first step. If our computations are being done in percentage changes, then when we reach box 6 with $r = \rho$, we can compute the updated flows according to

$$(Flow)_{\rho,s} = (Flow)_{\rho-1,s}(1 + 0.01p_{\rho,r})(1 + 0.01x_{\rho,s}) \ , \qquad \text{(E3.8.5)}$$

where $p_{\rho,s}$ and $x_{\rho,s}$ are the percentage changes in the ρ^{th} step in the relevant price and quantity. We can also use this formula when the computations are done in changes in the variables. However, we need an extra set of computations to get from results for changes in prices and quantities to the percentage changes required in (E3.8.5). If the computations are done in log changes, then a convenient updating formula is

$$(Flow)_{\rho,s} = exp[ln(Flow)_{\rho-1,s} + p_{\rho,s} + x_{\rho,s}] \qquad \text{(E3.8.6)}$$

where $p_{\rho,s}$ and $x_{\rho,s}$ are log changes computed in step ρ.

Once the input-output data have been updated, we return to box 2. There, the A matrix is reevaluated using cost and sales shares computed from the updated input-output flows. Thus, at each step, the coefficients in the A matrix incorporate the effects on cost and sales shares of changes in prices and quantities taking place at previous steps.

The computations continue until eventually we pass through box 4 with r+1 = s. At this stage we have completed s steps. We can now

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15 In large models we never evaluate the whole of B. By working with condensed systems we can limit our computations to selected rows. By applying (E3.8.4) for a subset of js we can limit our computations to selected columns.

compute the s-step estimates of the values reached by the endogenous variables given the total shocks in the exogenous variables. The relevant formulae for endogenous variable k are

$$(V_k)_{s,s} = (V_k)_{0,s} + \sum_{\rho=1}^{s} (v_k)_{\rho,s} \qquad \text{(E.3.8.7)}$$

when the v_ks are changes,

$$(V_k)_{s,s} = (V_k)_{0,s} \prod_{\rho=1}^{s} \left(1 + 0.01(v_k)_{\rho,s}\right) \qquad \text{(E.3.8.8)}$$

when the v_ks are percentage changes, and

$$(V_k)_{s,s} = exp\left[ln(V_k)_{0,s} + \sum_{\rho=1}^{s} (v_k)_{\rho,s}\right] \qquad \text{(E.3.8.9)}$$

when the v_ks are log changes. Rather than reporting the levels $(V_k)_{s,s}$, it is normally of more interest to report the percentage effects on the endogenous variables of the changes in the exogenous variables. The s-step estimates of these percentage effects can be computed as

$$100\left((V_k)_{s,s} - (V_k)_{0,s}\right)/ (V_k)_{0,s} \; .$$

(a) Use a sequence of calculations of the type outlined in Figure E3.8.1 to provide a two-step solution for the Stylized Johansen model. Assume that the initial situation is that depicted in Table E3.3.1. Assume that the exogenous variables are P_3 and X_4. Compute the effects of a 50 per cent increase (from 1 to 1.5) in the wage rate, P_3, holding constant the capital stock, X_4.

Hint: You will need only a pocket calculator if you work with log changes and use the information in Table E3.6.1. So that you can compare each stage of your calculations with ours, we suggest that you implement the 50 per cent increase in P_3 as two increases of $\frac{1}{2}ln(1.5)$ in $ln(P_3)$, i.e, put $(p_3)_{1,2} = (p_3)_{2,2} = \frac{1}{2}ln(1.5)$.

(b) What is the true solution for the effects on the endogenous variables of a 50 per cent increase in P_3? Can you write down the solution functions? That is, can you express Y, X_{10}, X_{20}, etc. as functions of P_3 and X_4?

Answer to Exercise 3.8

(a) Following the procedure outlined in Figure E3.8.1, we start by setting r at zero. The first arithmetic operation (box 2) is the evaluation of $A((V)_{0,s})$. This has been done in Exercise 3.4 and the answer is displayed in Table E3.4.2. Moving on to box 3, we accept the hint and set

$$(v_\beta)_{1,2} = \begin{bmatrix} p_3 \\ \\ x_4 \end{bmatrix}_{1,2} = \begin{bmatrix} \frac{1}{2}ln(1.5) \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 0.20273 \\ 0 \end{bmatrix}. \qquad \text{(E.3.8.10)}$$

Most of the computation in box 4 was completed in Exercise 3.6 where $B((V)_{0,2})$ was computed for the relevant closure (P_3, X_4 exogenous) and displayed in the right panel of Table E3.6.1. The vector $(v_\alpha)_{1,2}$ can be evaluated simply by multiplying the P_3-column of Table E3.6.1 by 0.20273. This gives

$$(v_\alpha)_{1,2} = (y, x_{10}, x_{20}, x_{11}, x_{21}, x_{31},$$
$$x_{41}, x_{12}, x_{22}, x_{32}, x_{42}, x_1,$$
$$x_2, x_3, p_1, p_2, p_4)_{1,2}$$

$$= (-0.30410, -0.30410, -0.35478, -0.30410, -0.35478, -0.50683,$$
$$0, -0.30410, -0.35478, -0.50683, 0, -0.30410,$$
$$-0.35478, -0.50683, 0, 0.05068, -0.30410).$$

Since s = 2 and r+1 is currently at 1, we move to box 5. There, r is increased to 2 taking us through to box 6. In box 6 we reevaluate the initial input-output flows from Table E3.3.1 according to formula (E3.8.6). Thus, for example, we have

$$\begin{aligned}(\text{Flow 1 to 1})_{1,2} &= exp[ln(\text{Flow 1 to 1})_{0,2} + (p_1)_{1,2} + (x_{11})_{1,2}] \\ &= exp[ln(4) + 0 - 0.30410] = 2.9511.\end{aligned}$$

The complete set of updated flows is in Table E3.8.1.

It is apparent that in generating Table E3.8.1, we have deflated each flow in Table E3.3.1 by the same percentage. Thus, in this particular example, when we return to box 2, we find that $A((V)_{1,2})$ is the same as the initial A matrix displayed in Table E3.4.2. This is because the elements of A are either ratios of flows (cost and sales shares) or constants. In box 3, we set p_3 and x_4 at the same values as they had in previous step, i.e., 0.20273 and 0. Since we arrive at box 4 with the same A matrix and v_β vector as in previous step, we emerge with the same v_α, that is

$$(v_\alpha)_{2,2} = (v_\alpha)_{1,2}.$$

Table E3.8.1

Input-Output Data after 1 Update; (Flow i to j)$_{1,2}$ in Dollars

		Industry 1	Industry 2	Households	Total Sales
Commodity	1	2.9511	1.4756	1.4756	5.9022
	2	1.4756	4.4267	2.9511	8.8534
Primary Factors	3	0.7378	2.2134		2.9511
	4	0.7378	0.7378		1.4756
Production		5.9022	8.8534	4.4267	

With r+1 at 2, we move to box 7. The two-step estimates of the values of the endogenous variables after a 50 per cent increase in P_3 can now be computed using (E3.8.9). For the first variable, household expenditure, we obtain

$$(Y)_{2,2} = exp[ln(Y_{0,2}) + y_{1,2} + y_{2,2}]$$
$$= exp[ln(6) - 0.30410 - 0.30410] = 3.2660.$$

Thus, our two-step estimate is that a 50 per cent increase in p_3 will reduce household expenditure by 45.57 per cent. Similarly we find that there are reductions of 45.57 per cent in X_{10}, X_{11}, X_{12}, X_1, and P_4. There are reductions of 50.81 per cent in X_{20}, X_{21}, X_{22}, and X_2. For X_{31}, X_{32} and X_3 the reductions are 63.71 per cent. P_2 increases by 10.67 per cent and there are no changes in X_{41}, X_{42} and P_1.

(b) Apart from rounding errors, the two-step solution obtained in part (a) is the true solution. One way of checking this is by substitution back into the structural form (E3.1.9), (E3.1.10), (E3.1.12), (E3.1.6), (E3.1.7) and (E3.1.23). For example, consider the household demand equations (E3.1.9). With i = 1, we have

$$LHS = (X_{10})_{2,2} = (0.5443)(X_{10})_{0,2} = (0.5443)(2) = 1.0886$$

and

$$RHS = \alpha_{10}(Y)_{2,2}/(P_1)_{2,2} = (0.\underline{3})(0.5443)(6)/1 = 1.0886 \quad .$$

In this particular example, substitution back into the structural equations may not be the cleverest way of establishing that the two-step solution is free from linearization error. Nevertheless, it is illustrative of the method that is available in most models for checking the validity

Table E3.8.2

Input-Output Data after 2 Updates; (Flow i to j)$_{2,2}$ in Dollars †

		Industry 1	Industry 2	Households	Total Sales
Commodity	1	2.1773	1.0887	1.0887	4.3546
	2	1.0887	3.2660	2.1773	6.5320
Primary Factors	3	0.5443	1.6330		2.1773
	4	0.5443	0.5443		1.0887
Production		4.3546	6.5320	3.2660	

†These flows can be computed using (E3.8.6) with $\rho = s = 2$.

of a suggested solution. In a few very large models, it may be too cumbersome to substitute into the left and right hand sides of every structural equation. In such cases, a useful minimum check is provided by the post-solution input-output table, i.e., the table of flows implied by the suggested solution. To obtain this table, we can make an extra update of the input-output flows by carrying out the computations in box 6 of Figure E3.8.1 with r = s. (The post-solution input-output flows for the computations in part (a) are given in Table E3.8.2.) Violations of the row and column sum balancing conditions in the post-solution input-output table would imply that the suggested solution is inconsistent with the structural equations requiring that for each industry the value of inputs equals the value of output and for each commodity the value of output equals the value of sales.

Normally, substitution of an s-step solution into the structural equations would produce discrepancies between left and right hand sides beyond what could be explained by rounding errors. We would also expect there to be differences between the i^{th} row and i^{th} column sums of the post-solution input-output table. We would be satisfied with the s-step solution if we judged the various discrepancies to be sufficiently small. In our Stylized model, however, we find that multi-step solutions computed with log changes produce no non-rounding discrepancies. This indicates that the solution equations are log linear. They are, in fact,

$$Y = C_1 X_4 P_3^{-1.5}, X_{10} = C_2 X_4 P_3^{-1.5}, \ldots, P_4 = C_{17} P_3^{-1.5},$$

where the exponents on the right hand sides have been taken from the right panel of Table E3.6.1 and the C_is are constants whose values can be determined from the initial data in Table E3.3.1. For example, C_1 is 6/2.

C. ON DERIVING PERCENTAGE-CHANGE FORMS

The problems in this section provide practice in deriving percentage- or log-change[16] forms for demand and supply systems associated with a variety of production and utility functions and production possibilities frontiers. By the time you finish these problems, we hope that you will feel confident about deriving percentage-change forms for any of the specifications you are likely to want to use in practice.

Exercise 3.9 Linearizing the input demand functions from a CES production function[17]

Assume that a firm facing given input prices, $P_1, \ldots, P_n$, chooses input levels, $X_1, \ldots, X_n$, so they minimize the cost, $\Sigma_i P_i X_i$, of producing a given output, Y, subject to the CES (constant elasticity of substitution) production function

$$Y = A\left[\sum_{i=1}^{n} \delta_i X_i^{-\rho}\right]^{-1/\rho} \tag{E3.9.1}$$

where A and the δ_is are positive parameters with $\Sigma_i \delta_i = 1$ and ρ is a parameter whose value is greater than or equal to -1 but not equal to zero.[18]

Derive the percentage-change form for the input demand functions. Avoid corner solutions by assuming that $\rho > -1$.

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16 Percentage-change and log-change forms are identical. For expositional simplicity we refer in the remainder of this section to percentage changes only.

17 The CES production function was first applied by Arrow, Chenery, Minhas and Solow (1961). For an exercise which develops the properties of the CES production function in detail, see Dixon, Bowles and Kendrick (1980, Exercise 4.20).

18 As ρ approaches zero, (E3.9.1) approaches a Cobb-Douglas form, see Dixon, Bowles and Kendrick (1980, E4.20).

Answer to Exercise 3.9

The first-order conditions for cost minimization are that there exists Λ such that Λ and the X_ks jointly satisfy

$$P_k = \Lambda A \left[\sum_{i=1}^{n} \delta_i X_i^{-\rho} \right]^{-(1+\rho)/\rho} \delta_k X_k^{-(1+\rho)}, \; k = 1, \ldots, n \qquad \text{(E3.9.2)}$$

and

$$Y = A \left[\sum_{i=1}^{n} \delta_i X_i^{-\rho} \right]^{-1/\rho} . \qquad \text{(E3.9.3)}$$

By using (E3.9.3), we can replace (E3.9.2) with the more convenient equations

$$P_k = \Lambda \, A^{-\rho} \, Y^{(1+\rho)} \, \delta_k X_k^{-(1+\rho)}, \; k = 1, \ldots, n. \qquad \text{(E3.9.4)}$$

In percentage change form (E3.9.4) and (E3.9.3) can be written as

$$p_k = \lambda + (1+\rho)y - (1+\rho)x_k \qquad \text{(E3.9.5)}$$

and

$$y = \sum_k S_k x_k \qquad \text{(E3.9.6)}$$

where p_k, λ, y and x_k are percentage changes in P_k, Λ, Y and X_k, and

$$S_k = \delta_k X_k^{-\rho} / \left(\sum_i \delta_i X_i^{-\rho} \right) \quad \text{for all } k. \qquad \text{(E3.9.7)}$$

Equation (E3.9.4) implies that

$$P_k X_k / \sum_i P_i X_i = \delta_k X_k^{-\rho} / \sum_i \delta_i X_i^{-\rho} \quad . \qquad \text{(E3.9.8)}$$

Thus, S_k is the share of input k in total costs.

From (E3.9.5) we find that

$$x_k = -\sigma p_k + \sigma\lambda + y \qquad \text{(E3.9.9)}$$

where σ is the positive parameter defined by

$$\sigma = 1 / (1+\rho) \; . \qquad \text{(E3.9.10)}$$

Substitution from (E3.9.9) into (E3.9.6) gives

$$y = -\sigma \sum_k S_k p_k + \sigma\lambda + y$$

leading to

$$\lambda = \sum_k S_k p_k \; . \qquad \text{(E3.9.11)}$$

Now we substitute from (E3.9.11) into (E3.9.9) to obtain the percentage change form for the input demand functions:

$$x_k = y - \sigma\left(p_k - \sum_{i=1}^{n} S_i p_i\right) \quad \text{for } k = 1, \ldots, n. \qquad \text{(E3.9.12)}$$

Equation (E3.9.12) says that in the absence of price changes, all input volumes move by the same percentage as output. This reflects the constancy of the returns to scale exhibited by the production function (E3.9.1). If the price of input k rises relative to a cost-share weighted index of all input prices, then the use of input k will fall relative to output (i.e., X_k/Y will decline). There will be substitution away from input k. The strength of this substitution effect depends on the size of the parameter σ, which is the elasticity of substitution between any pair of inputs.

It is worth noting that in our derivation of the percentage change form, (E3.9.12), we worked with a percentage change version of the first-order conditions (E3.9.2) and (E3.9.3). This approach is usually easier than the alternative where the input demand functions are first derived and then linearized.

Exercise 3.10 Linearizing the input demand functions from a CRESH production function[19]

Assume that the production function has the CRESH (constant ratios of elasticities of substitution, homothetic) form, i.e.,

$$\sum_{i=1}^{n} \left[\frac{X_i}{Y}\right]^{h_i} \frac{Q_i}{h_i} = \alpha \qquad \text{(E3.10.1)}$$

where Y is output, the X_is are inputs and the Q_is, h_is and α are parameters. Each h_i is less than 1 but not equal to zero. Each Q_i is positive and the Q_is and α are normalized so that $\sum_i Q_i = 1$. In general, α can have either sign but if each of the Q_i/h_i has the same sign, then α must have their common sign. As in Exercise 3.9, derive a percentage-change form for the input demand functions assuming that the firm treats input prices, P_k, $k = 1, \ldots, n$, as beyond its control and chooses its input levels to minimize the cost of producing any given level of output.

Answer to Exercise 3.10

The first-order conditions for cost minimization are that there exists Λ such that Λ and the X_ks jointly satisfy

$$P_k = \Lambda\left[\frac{X_k^{h_k-1}}{Y^{h_k}}\right] Q_k\ , \qquad k = 1, \ldots, n \qquad \text{(E3.10.2)}$$

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19 CRESH functions were introduced by Hanoch (1971). For an exercise which develops the properties of the CRESH production function in detail, see Dixon, Bowles and Kendrick (1980, Exercise 4.21).

and

$$\sum_{i=1}^{n}\left[\frac{X_i}{Y}\right]^{h_i}\frac{Q_i}{h_i} = \alpha\ . \tag{E3.10.3}$$

In percentage change form (E3.10.2) and (E3.10.3) can be written as

$$p_k = \lambda + (h_k - 1)x_k - h_k y, \qquad k = 1, \ldots, n \tag{E3.10.4}$$

and

$$\sum_{i=1}^{n} h_i(x_i - y)W_i = 0 \tag{E3.10.5}$$

where p_k, λ, y and x_k are percentage changes in P_k, Λ, Y and X_k, and

$$W_i = \left[\frac{X_i}{Y_i}\right]^{h_i}\frac{Q_i}{h_i}, \qquad i = 1, \ldots, n. \tag{E3.10.6}$$

By multiplying (E3.10.2) through by X_k, we can show that

$$h_k W_k \ / \ \sum_{i=1}^{n} h_i W_i \quad = \quad S_k, \qquad k = 1, \ldots, n,$$

where S_k is the share of input k in total costs. Hence, (E3.10.5) may be rewritten as

$$\sum_{k=1}^{n} S_k x_k = y. \tag{E3.10.7}$$

Next, we rearrange (E3.10.4) as

$$x_k = \left[\frac{1}{h_k - 1}\right](p_k - \lambda + h_k y). \tag{E3.10.8}$$

Then by substitution into (E3.10.7) we find that

$$y = \sum_k \frac{S_k}{(h_k - 1)}(p_k - \lambda + h_k y)\ . \tag{E3.10.9}$$

Hence,

$$\lambda \ = \ y + \sum_k S_k^* p_k \tag{E3.10.10}$$

where S_k^* is the modified cost share defined by

$$S_k^* = \frac{S_k/(1-h_k)}{\sum_i S_i/(1-h_i)}\ . \tag{E3.10.11}$$

Now we substitute from (E3.10.10) into (E3.10.8) to obtain the percentage-change form for the input demand functions as

$$x_k = y - \sigma_k(p_k - \Sigma_i S_i^* p_i) , \qquad k = 1, \dots, n, \tag{E3.10.12}$$

where σ_k is the positive parameter defined by

$$\sigma_k = 1/(1-h_k). \tag{E3.10.13}$$

Equation (E3.10.12) differs from (E3.9.12), which we derived for the CES case, in two respects. First, the weights used in computing the average movement in the input prices are 'modified' cost shares rather than cost shares. Second, (E3.10.12) generalizes (E3.9.12) by allowing the coefficient, σ_k, on the relative price term to vary across inputs.

Exercise 3.11 Supply response functions with CET and CRETH transformation frontiers[20]

Assume that a firm facing given prices, $P_1, \dots, P_m$, for its m products chooses its output levels, $Y_1, \dots, Y_m$, to maximize total revenue, $\Sigma_i P_i Y_i$, subject to the CET (constant elasticity of transformation) production possibilities frontier

$$Z = B\left[\sum_{i=1}^{m} \gamma_i Y_i^{-\rho}\right]^{-1/\rho} \tag{E3.11.1}$$

where B and the γ's are positive parameters with $\Sigma_i \gamma_i = 1$ and ρ is a parameter whose value is less than or equal to -1.[21] Z is a measure of the firm's overall capacity to produce or activity level. The value of Z depends on the quantities of inputs. In the present problem, where we are determining the composition of the firm's output, we will treat Z as an exogenous variable.

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20 CET functions were first applied by Powell and Gruen (1967 and 1968). CRETH functions were first applied by Vincent, Dixon and Powell (1980). CRETH functions are used in modeling the agricultural sector in the *ORANI* model of the Australian economy, see Dixon, Parmenter, Sutton and Vincent (1982, pp. 68-94 and 191-194).

21 The CET form is identical to the CES form apart from the restriction on ρ. With CES, ρ is greater than or equal to -1; with CET, ρ is less than or equal to -1. In the CES case, the contours are concave from above. In the CET case, the contours are concave from below.

(a) Sketch the production possibilities frontier assuming that m = 2, $\gamma_1 = \gamma_2 = 1/2$, B = 1, Z = 1 and $\rho = -2$.

(b) What happens to the production possibilities frontier as Z changes?

(c) Assume that m = 2, $\gamma_1 = \gamma_2 = 1/2$, B = 1 and Z = 1. Describe how the shape of the production possibilities frontier changes as ρ moves from −1 towards negative infinity.

(d) In this problem we are assuming that the product composition of the firm's output can be determined independently of the composition of the firm's inputs. Describe the circumstances under which such an assumption would be appropriate.

(e) Derive percentage-change forms for the firm's supply response functions; i.e., relate the percentage changes, y_k, k = 1, ..., m, in output levels to the percentage change, z, in total capacity or activity and the percentage changes, p_k, k = 1, ..., m, in the product prices. To avoid corner solutions, assume that $\rho < -1$.

(f) Replace the CET function (E3.11.1) by the more general CRETH (constant ratios of elasticities of transformation, homothetic) function,

$$\sum_{i=1}^{m} \left[\frac{Y_i}{Z}\right]^{h_i} \frac{V_i}{h_i} = \beta, \qquad \text{(E3.11.2)}$$

where the V_is, h_is and β are parameters. Each h_i is greater than 1, while β and each of the V_is is positive with $\Sigma_i V_i = 1$.[22] Derive a percentage-change form for the supply response functions.

Answer to Exercise 3.11

(a) With the parameters set at the given values, we have

$$2 = Y_1^2 + Y_2^2 . \qquad \text{(E3.11.3)}$$

Assuming that only nonnegative values are allowed for the Y_is, the production possibilities frontier is the quarter circle, ATB, shown in Figure E3.11.1.

(b) Changes in Z produce radial expansions and contractions of the production possibilities frontier. If we increase input levels sufficiently

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22 The CRETH form is identical to the CRESH form (see Exercise 3.10) apart from the restrictions on the h_is.

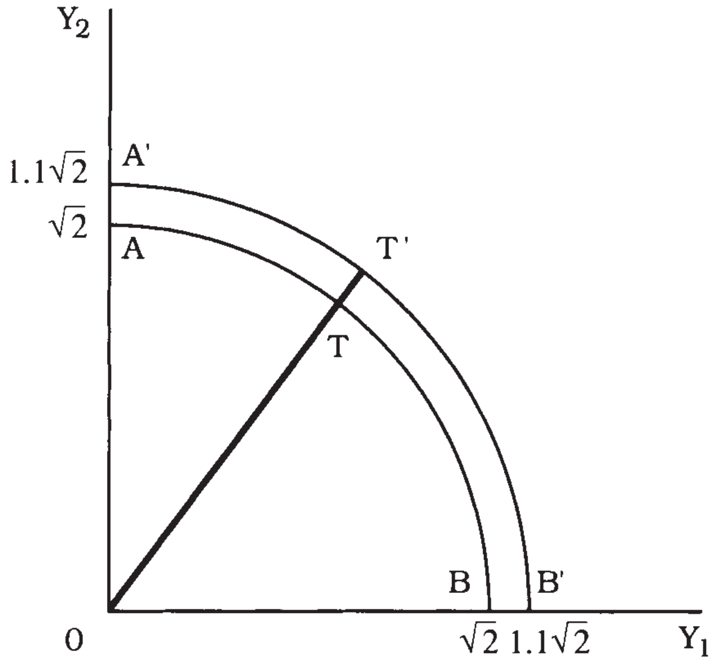

Figure E3.11.1 The quarter circle production possibilities frontier

ATB is the production possibilities frontier for the special case of (E3.11.1) where $m = 2$, $\gamma_1 = \gamma_2 = 0.5$, $B = 1$, $Z = 1$ and $\rho = -2$. A'T'B' is the production possibilities frontier after Z has been increased by 10 per cent.

to increase Z by, for example, 10 per cent, then we can increase the output of each commodity by 10 per cent. All points on the new production possibilities frontier can be obtained by drawing rays (e.g., OT in Figure E3.11.1) from the origin to the initial production possibilities frontier and then extending their lengths by 10 per cent (to T').

(c) With the given parameter values, the production possibilities frontier, (E3.11.1), is

$$Z = [\tfrac{1}{2}Y_1^{-\rho} + \tfrac{1}{2}Y_2^{-\rho}]^{-1/\rho} \ . \tag{E3.11.4}$$

Its slope is given by[23]

$$Slope = -\frac{\partial Z/\partial Y_1}{\partial Z/\partial Y_2} = -(Y_1/Y_2)^{-\rho-1} \ . \tag{E3.11.5}$$

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23 When the axes are labeled as in Figure E3.11.1, the slope of the production possibilities frontier is (apart from sign) the marginal rate of transformation of good 1 into good 2, i.e., the rate of increase in the output of good 2 made possible per unit reduction in the output of good 1.

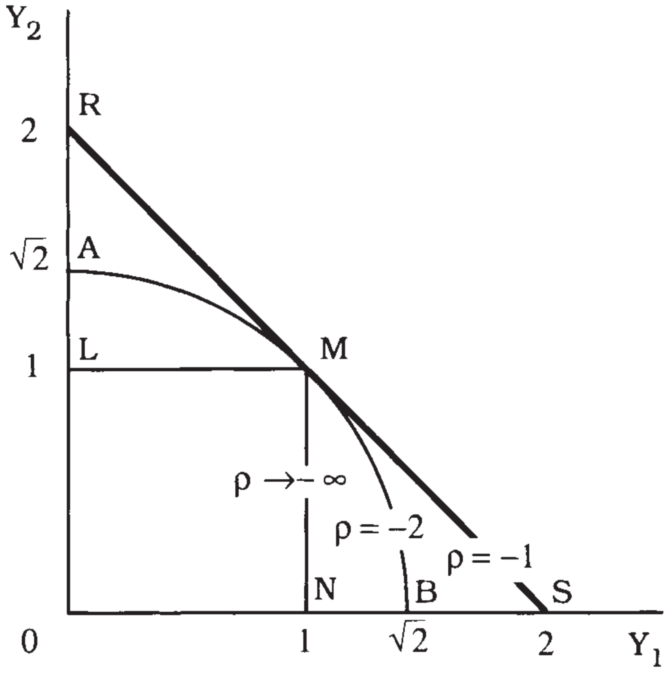

Figure E3.11.2 Production possibilities frontiers given by (E3.11.4) when $\rho = -1$, -2 and $-\infty$

RMS is the production possibilities frontier given by (E3.11.4) when $\rho = -1$. As ρ approaches negative infinity, the production possibilities frontier approaches LMN. The quarter circle AMB is reproduced from Figure E3.11.1 for the case $\rho = -2$.

If $\rho = -1$, then (E3.11.4) reduces to

$$Z = \frac{1}{2}Y_1 + \frac{1}{2}Y_2$$

and the production possibilities frontier is a straight line. In this case, goods 1 and 2 are perfectly transformable. If ρ approaches negative infinity, then at points where Y_1/Y_2 is less than one, (E3.11.5) implies that the slope of the production possibilities frontier approaches zero. At points where Y_1/Y_2 is greater than one, the slope approaches negative infinity. Thus, with $Z = 1$, the production possibility frontier moves closer and closer to LMN in Figure E3.11.2 as ρ approaches negative infinity. In the extreme case, goods 1 and 2 will be produced in fixed proportions and revenue maximizing production will be at point M irrespective of commodity prices.

(d) The assumption is appropriate only if the inputs are of a general-purpose nature. For example, in modeling the agricultural production of a particular region, we could assume that the use of labor, farm machinery, fertilizer and land gives the region a capacity (measured by Z) to produce. We might describe the creation of this capacity by

$$Z = f(L, K, F, N)$$

where f is a concave function[24] (perhaps of the CES or CRESH variety studied in Exercises 3.9 and 3.10) and L, K, F and N are inputs of labor, capital, fertilizer and land. Because these inputs are of a general-purpose nature, we might assume that the capacity to produce can be used to generate a variety of combinations of, say, wheat and wool, with the possible combinations being described by (E3.11.1) or some other convex function.[25] On the other hand, if silo space (wheat-specific) and shearing (wool-specific) were included among our inputs, then we could not separate the determination of the input-mix from that of the output-mix. The input and output mixes would need to be considered simultaneously.

(e) The first-order conditions for revenue maximization are that there exists Λ such that Λ and the Y_ks jointly satisfy

$$P_k = \Lambda B \left[\sum_{i=1}^{m} \gamma_i Y_i^{-\rho} \right]^{-(1+\rho)/\rho} \gamma_k Y_k^{-(1+\rho)} , \qquad k = 1, \dots, m \quad \text{(E3.11.6)}$$

and

$$Z = B \left[\sum_{i=1}^{m} \gamma_i Y_i^{-\rho} \right]^{-1/\rho} . \qquad \text{(E3.11.7)}$$

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24 f is a concave function on the convex set $S \subset \mathbf{R}^n$ if and only if

$$f[\alpha x + (1-\alpha)y] \geq \alpha f(x) + (1-\alpha)f(y) \text{ for all } \alpha \in [0,1] \text{ and } x,y \in S$$

(see Katzner (1970, p. 183)). The usual single-output production functions exhibiting constant or diminishing returns to scale and having isoquants of the familiar shape are concave on the positive orthant.

25 f is a convex function on the convex set $S \subset \mathbf{R}^n$ if and only if

$$f[\alpha x + (1-\alpha)y] \leq \alpha f(x) + (1-\alpha)f(y) \text{ for all } \alpha \in [0,1] \text{ and } x,y \in S$$

(see Katzner (1970, p.183)). Production possibilities functions exhibiting constant or diminishing returns to scale with production possibilities frontiers of the usual shape are convex on the positive orthant. Note that a production possibilities function exhibits diminishing returns to scale if a 1 per cent (say) increase in the output of all commodities requires a greater than one per cent increase in capacity (Z in this exercise). By contrast, a production function exhibits diminishing returns to scale if a 1 per cent increase in all inputs produces less than a one per cent increase in capacity.

By repeating the steps that took us from (E3.9.2) and (E3.9.3) to (E3.9.12), we can go from (E3.11.6) and (E3.11.7) to the percentage-change form

$$y_k = z - \theta(p_k - \textstyle\sum_i R_i p_i) , \quad k = 1, ..., m, \qquad \text{(E3.11.8)}$$

where θ is the negative[26] parameter defined by $\theta = 1/(1+\rho)$ and the Rs are revenue shares defined by $R_k = P_k Y_k / \sum_i P_i Y_i$, for all k.

The interpretation of (E3.11.8) is similar to that of (E3.9.12). In the absence of price changes, all output volumes move by the same percentage as overall capacity. This reflects the constancy of the returns to scale exhibited by the production possibilities function (E3.11.1). If the price of product k rises relative to a revenue-share weighted index of all product prices, then the output of k will rise relative to productive capacity (i.e., Y_k/Z will increase). There will be be transformation towards product k. The strength of this transformation effect depends on the size of the parameter θ which is the elasticity of transformation between any pair of outputs.

(f) By following the method used in Exercise 3.10, we find that

$$y_k = z - \theta_k(p_k - \textstyle\sum_i R_i^* p_i) , \quad k = 1, ..., m, \qquad \text{(E3.11.9)}$$

where θ_k is the negative parameter defined by $\theta_k = 1/(1-h_k)$ and the R^*s are modified revenue shares defined by

$$R_k^* = \frac{R_k/(1-h_k)}{\sum_i R_i/(1-h_i)} ,$$

with the Rs being revenue shares.

Equation (E3.11.9) differs from the corresponding equation, (E3.11.8), for the CET case in the same ways as (E3.10.12) differs from (E3.9.12). First, the weights used in computing the average movement in output prices are 'modified' revenue shares rather than revenue shares. Second, (E3.11.9) generalizes (E3.11.8) by allowing the coefficient, θ_k, on the relative price term to vary across outputs.

Exercise 3.12 The translog unit cost function

The transcendental logarithmic or translog function is a convenient specification for unit cost functions and indirect utility functions. It underlies much of the econometric work by Dale

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26 Remember that $\rho < -1$.

Jorgenson and others[27] on systems of sectoral input demand equations and household consumption equations. Of particular importance to the further development of applied general equilibrium economics as an empirical field is Jorgenson's (1984) paper "Econometric methods for applied general equilibrium modeling" in which nested translog unit cost and indirect utility functions are estimated at a disaggregated level with U.S. data. In this exercise, we ask you to review the properties of the translog unit cost function.[28]

The translog unit cost function has the form

$$lnQ(P) \quad = \quad A + \sum_i B_i lnP_i + \frac{1}{2}\sum_i \sum_j C_{ij}(lnP_i)(lnP_j) \quad , \qquad \text{(E3.12.1)}$$

where $Q(P)$ is the cost per unit of output[29] when the input prices are $P' = (P_1, P_2, \ldots, P_n)$[30] and A, B_i and C_{ij} are parameters with $C_{ij} = C_{ji}$ for all $i \neq j$.[31]

(a) What restrictions should the B_is and C_{ij}s satisfy to ensure that Q is homogeneous of degree one with respect to input prices?

(b) Assuming that (E3.12.1) is a legitimate description of the relationship between unit costs and input prices, derive a percentage-change form for the input demand functions convenient for use in a Johansen model.

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27 Early applications of translog functions include Berndt and Jorgenson (1973), and Berndt and Wood (1975). See also Hudson and Jorgenson (1974) where translog unit cost functions form part of a general equilibrium model of U.S. energy usage. Recent work is reported in Jorgenson (1984).

28 For an exercise on the properties of cost functions in general, see Dixon, Bowles and Kendrick (1980, Exercise 4.17).

29 We assume that the underlying production function exhibits constant returns to scale. Otherwise, unit costs would not be independent of the output level. See Exercise 3.1(e).

30 We assume that all prices are greater than zero. The RHS of (E3.12.1) is not defined unless $P > 0$.

31 Let C be the $n \times n$ matrix of C_{ij}s. If C were not symmetric, then we could simply rewrite (E3.12.1) with our initial C matrix replaced by the symmetric matrix $\frac{1}{2}(C+C')$. Therefore, no loss of generality is incurred by assuming that we have chosen a symmetric C matrix.

(c) In estimating the parameters of (E3.12.1) we can use the fact that cost functions are concave with respect to input prices.[32] What parameter restrictions are suggested by concavity?

Hint: Part (c) is rather difficult. We have provided a detailed answer which we hope will help you in reading Jorgenson (1984).

Answer to Exercise 3.12

(a) We require that

$$Q(\lambda P) = \lambda Q(P) \quad \text{for all } P > 0 \text{ and } \lambda > 0. \tag{E3.12.2}$$

Equivalently, we require that

$$lnQ(\lambda P) = lnQ(P) + ln\lambda \quad \text{for all } P > 0 \text{ and } \lambda > 0. \tag{E3.12.3}$$

From (E3.12.1) we have

$$lnQ(\lambda P) = A + \sum_i B_i ln(\lambda P_i) + \frac{1}{2}\sum_i\sum_j C_{ij}\left([ln(\lambda P_i)][ln(\lambda P_j)]\right). \tag{E3.12.4}$$

Since $ln(\lambda P_k) = ln\lambda + lnP_k$ for all k and $C_{ij} = C_{ji}$ for all $i \neq j$, (E3.12.4) may be expanded as

$$lnQ(\lambda P) = lnQ(P) + (ln\lambda)\sum_i B_i + ln\lambda\sum_i(lnP_i)\sum_j C_{ij} + \frac{1}{2}(ln\lambda)^2\sum_i\sum_j C_{ij}\,. \tag{E3.12.5}$$

Hence, necessary and sufficient conditions for (E3.12.3) are

$$\sum_i B_i = 1 \tag{E3.12.6}$$

and

$$\sum_j C_{ij} = 0 \quad \text{for all } i. \tag{E3.12.7}$$

(b) We rewrite (E3.12.1) as

$$Q(P) = exp\left(A + \sum_i B_i lnP_i + \frac{1}{2}\sum_i\sum_j C_{ij}(lnP_i)(lnP_j)\right). \tag{E3.12.8}$$

From Shepard's lemma (see for example, Dixon, Bowles and Kendrick (1980, Exercise 4.17)), we know that the input demands are the derivatives

$$X_k = Y\frac{\partial Q(P)}{\partial P_k} \quad \text{for all } k, \tag{E3.12.9}$$

where Y is the level of output and X_k is the demand for input k. Thus, from (E3.12.8), we have

$$X_k = YQ(P)\left(B_k + \sum_j C_{kj}(lnP_j)\right)/\,P_k \quad \text{for all } k. \tag{E3.12.10}$$

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32 If you have forgotten why cost functions are concave with respect to input prices, then you should look at Dixon, Bowles and Kendrick (1980, Exercise 4.17).

In percentage-change form (E3.12.10) becomes

$$x_k = y + q + \sum_j \left[\frac{C_{kj}}{B_k + \sum_t C_{kt}\,(lnP_t)} \right] p_j - p_k \qquad \text{for all k,} \quad \text{(E3.12.11)}$$

where, as usual, the lower case symbols, x, y, q and p represent percentage changes in the variables denoted by the corresponding upper case symbols. To turn (E3.12.11) into the required input demand functions we must eliminate q. We start by noting that

$$q = \sum_k \left[\frac{\partial Q(P)}{\partial P_k} \frac{P_k}{Q(P)} \right] p_k \,. \qquad \text{(E3.12.12)}$$

It follows easily from Shepard's lemma, (E3.12.9), that the elasticities of the unit cost function are the input shares in total costs, i.e.,

$$\frac{\partial Q(P)}{\partial P_k} \frac{P_k}{Q(P)} = S_k \qquad \text{for all k,} \qquad \text{(E3.12.13)}$$

where $S_k = P_k X_k \,/\, \big(YQ(P)\big)$. Hence (E3.12.12) may be rewritten as

$$q = \sum_k S_k p_k \,. \qquad \text{(E3.12.14)}$$

We also note from (E3.12.10) that in the case of translog unit cost functions, the cost shares are given by

$$S_k = B_k + \sum_j C_{kj}(lnP_j) \qquad \text{for all k.} \qquad \text{(E3.12.15)}$$

On substituting from (E3.12.14) and (E3.12.15) into (E3.12.11) we obtain

$$x_k = y - \big(p_k - \sum_j S_{kj} p_j\big) \qquad \text{for all k,} \qquad \text{(E3.12.16)}$$

where the S_{kj}s are modified cost shares[33] defined by

$$S_{kj} = S_j + (C_{kj} \,/\, S_k) \qquad \text{for all k and j.}$$

Equation (E3.12.16) sets out the system of input demand equations in a form convenient for use in a Johansen model. It expresses percentage changes in the input demands as linear functions of percentage changes in the input prices and output. The coefficients are easily calculated modified cost shares. As in (E3.9.12) and (E3.10.12), the percentage change in the demand for input k is the difference between an activity term and a substitution term. In the

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33 Note that $\sum_j S_{kj} = \sum_j S_j + \sum_j (C_{kj} \,/\, S_k) = 1$ for all k.

absence of changes in relative prices, the volume of each input increases by the same percentage as output reflecting constant returns to scale in the underlying production function. There will be substitution away from input k if the price of input k rises relative to a weighted average of the percentage changes in all input prices. Unlike the weights appearing in (E3.9.12) and (E3.10.12), the weights in (E3.12.16) need not all be positive. The translog unit cost function allows for complementarity between inputs. If C_{kj} is sufficiently negative, then S_{kj} will be negative and an increase in P_j (with all other prices and output held constant) will reduce the demand for input k.

(c) There are two ways to interpret the translog unit cost function, (E3.12.1). One is to think of it as a second-order Taylor's-series approximation to the true unit cost function. On this interpretation, $Q(P)$ need exhibit concavity only in the neighborhood of a central price vector, $\bar{P}$, possibly a vector of sample means. The second interpretation is that $Q(P)$ is, itself, the true unit cost function. On this interpretation, the parameters of (E3.12.1) should be restricted to ensure concavity, if not globally, then at least over a large subset of the price space.

Consider, first, the Taylor's-series interpretation. Where $\Gamma(P)$ is the true unit cost function, it may be written as

$$ln(\Gamma(P)) = ln\left[\Gamma\left(exp(lnP_1), \ldots, exp(lnP_n)\right)\right] \qquad \text{(E3.12.17)}$$

or

$$ln(\Gamma(P)) = g(lnP_1, lnP_2, \ldots, lnP_n) \ , \qquad \text{(E3.12.18)}$$

with g being defined by (E3.12.17) and (E3.12.18). It is convenient to choose quantity units so that the central prices are all unity, giving[34]

$$ln\bar{P} = 0.$$

Then, (E3.12.18) can be expanded as

$$ln(\Gamma(P)) = g(0) + \sum_i \left[\frac{\partial g(0)}{\partial lnP_i}\right] lnP_i + \frac{1}{2}\sum_i\sum_j \left[\frac{\partial^2 g(0)}{\partial lnP_i \, \partial lnP_j}\right] lnP_i \, lnP_j$$

$$+ \text{ higher order terms.}$$

Hence, under the Taylor's-series interpretation of (E3.12.1), we have

$$B_i = \frac{\partial g(0)}{\partial lnP_i} , \qquad i = 1, \ldots, n \qquad \text{(E3.12.19)}$$

34 We use the short-hand notation lnP to denote the vector $(lnP_1, lnP_2, \ldots, lnP_n)$.

and
$$C_{ij} = \frac{\partial^2 g(0)}{\partial lnP_i\, \partial lnP_j}, \quad i,j = 1, \ldots, n. \tag{E3.12.20}$$

That is, the B_is and C_{ij}s in (E3.12.1) are first- and second-order partial derivatives of the g function evaluated at a central point, $ln\bar{P} = 0$. Notice also that
$$\frac{\partial g(lnP)}{\partial lnP_i} = \frac{\partial ln(\Gamma(P))}{\partial lnP_i} = \frac{\partial \Gamma(P)}{\partial P_i}\frac{P_i}{\Gamma(P)} \tag{E3.12.21}$$
and that by Shepard's lemma
$$\partial \Gamma(P)/\partial P_i = X_i/Y \quad , \tag{E3.12.22}$$
where X_i is the demand for input i and Y is the level of output. Thus, we see from (E3.12.19) that B_i is the share of input i in total costs when $P = \bar{P}$, i.e.,
$$B_i = \bar{P}_i\bar{X}_i \,/\, \left(\Gamma(\bar{P})Y\right). \tag{E3.12.23}$$

Since $\Gamma(P)$ is concave, we know that its Hessian, H(P), (the matrix of second-order partial derivatives), must be negative semidefinite (see, for example, Katzner (1970, pp. 200-201).[35] In particular, $H(\bar{P})$ is negative semidefinite. Our task is to translate this condition on $H(\bar{P})$ into a condition on the parameters of (E3.12.1). First, we must express H(P) in terms of partial derivatives of g(*ln*P). Then we will be able to use (E3.12.19) and (E3.12.20) in translating the restriction on $H(\bar{P})$ into restrictions on C_{ij} and B_i.

Starting from (E3.12.21) we have
$$\frac{\partial \Gamma(P)}{\partial P_i} = \frac{\Gamma(P)}{P_i}\frac{\partial g(lnP)}{\partial lnP_i}. \tag{E3.12.24}$$

Thus the ij^th^ entry in the Hessian matrix, H(P), is given by
$$\frac{\partial^2 \Gamma(P)}{\partial P_i \partial P_j} = \frac{\partial \Gamma(P)}{\partial P_j}\frac{1}{P_i}\frac{\partial g}{\partial lnP_i} - \delta_{ij}\frac{\Gamma(P)}{P_i^2}\frac{\partial g}{\partial lnP_i} + \frac{\Gamma(P)}{P_iP_j}\left[\frac{\partial^2 g}{\partial lnP_i\, \partial lnP_j}\right]$$
$$\text{for } i,j = 1, \ldots, n, \tag{E3.12.25}$$
where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. On substituting from (E3.12.19), (E3.12.20) and (E3.12.21) into (E3.12.25) and recalling that $\bar{P}_i = 1$ for all i, we obtain
$$H(\bar{P}) = \Gamma(\bar{P})\,(C + BB' - \hat{B}), \tag{E3.12.26}$$
where C is the $n \times n$ matrix of C_{ij}s, B is the $n \times 1$ vector of B_is and $\hat{B}$ is the diagonal matrix formed from B.

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35 Nothing more follows from concavity. Negative semidefiniteness of H(P) is sufficient for concavity as well as necessary.

We assume that $\Gamma(\bar{P})$ is strictly positive. Hence, (E3.12.26) implies that $H(\bar{P})$ will be negative semidefinite if and only if $C + BB' - \hat{B}$ is negative semidefinite. Thus, in summary, the Taylor's-series interpretation of (E3.12.1) suggests that the B_is can be set equal to cost shares observed at a central point where all prices are unity and that C should be estimated as a symmetric matrix subject to the restrictions that

$$C\underline{1} = 0 \qquad \text{(E3.12.27)[36]}$$

and

$$C + BB' - \hat{B} \quad \text{is negative semidefinite.} \qquad \text{(E3.12.28)}$$

Now consider the second interpretation of (E3.12.1). Under this interpretation, the true unit cost function is

$$Q(P) \;=\; exp\left[A + \textstyle\sum_i B_i lnP_i + \frac{1}{2}\sum_i\sum_j C_{ij}(lnP_i)(lnP_j)\right] . \qquad \text{(E3.12.29)}$$

Thus we have, as we did in part (b),

$$\frac{\partial Q(P)}{\partial P_i} \;=\; \frac{Q(P)}{P_i}\,(B_i + \textstyle\sum_j C_{ij} lnP_j) \;. \qquad \text{(E3.12.30)}$$

Notice that $\partial Q(P)/\partial P_i$ cannot be nonnegative for all $P > 0$ unless $C_{ij} = 0$ for all i and j. Obviously we would not wish to restrict the C_{ij}s to zero. If we did so, we would be back to the Cobb-Douglas situation of fixed cost shares. If we had been happy with a Cobb-Douglas specification, we would not be worrying about translog functions. Thus, in using the translog function, we cannot insist on global monotonicity, i.e., we cannot insist that

$$\frac{\partial Q(P)}{\partial P_i} \;\geq\; 0 \qquad \text{for all } P > 0, \qquad i = 1, ..., n.$$

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36 In (E3.12.27) we have simply repeated restriction (E3.12.7) derived in part (a). However, the derivation in part (a) was based on the assumption that Q(P) is the true unit cost function, not just a second-order approximation. In the present context, we know that $\Gamma(P)$ is homogeneous of degree 1 with respect to prices. Therefore $\Gamma_i(P)$ is homogeneous of degree zero where we use the subscript i to denote partial differentiation with respect to P_i. Hence, from Euler's theorem, we have

$$\textstyle\sum_j \Gamma_{ij}(P)P_j = 0 \text{ for all i,}$$

where $\Gamma_{ij}(P)$ is the second-order partial derivative of $\Gamma(P)$ with respect to P_i and P_j. By substituting from (E3.12.25) and by recalling that the linear homogeneity of $\Gamma(P)$ implies that

$$\textstyle\sum_j \Gamma_j\,(P)\,P_j = \Gamma(P)\;,$$

we find that

$$\sum_j \frac{\partial^2 g(lnP)}{\partial lnP_i \partial lnP_j} \;=\; 0 \;\text{ for all i.}$$

In particular, this condition holds at $P = \bar{P}$, justifying (E3.12.27).

Consequently, we cannot interpret (E3.12.29) as a globally valid description of unit costs.

In his econometric estimation of the parameters of translog unit cost functions, Jorgenson (1984) interpreted (E3.12.29) as the true unit cost function for all $P > 0$ such that monotonicity is satisfied. In addition to the usual symmetry and homogeneity restrictions,

$$C' = C \ , \tag{E3.12.31}$$

$$\sum_i B_i = 1 \ , \tag{E3.12.32}$$

and

$$C\underline{1} = 0 \ , \tag{E3.12.33}$$

he imposed the restriction

$$C \text{ is negative semidefinite.} \tag{E3.12.34}$$

This ensures that the Hessian matrix of $Q(P)$, $H_Q(P)$, is negative semidefinite for all P in the set

$$L = \left\{ P \mid P > 0, \ \frac{\partial Q(P)}{\partial P_i} \geq 0, \ i = 1, \ldots, n \right\} . \tag{E3.12.35}$$

This, in turn, ensures that $Q(P)$ is concave over any convex set in L.[37]

To demonstrate the negative semidefiniteness of $H_Q(P)$ under (E3.12.31) - (E3.12.34), we start by differentiating in (E3.12.30) with respect to P_j to obtain the components of $H_Q(P)$ as

$$\frac{\partial^2 Q}{\partial P_i \partial P_j} = \frac{Q(P)}{P_i P_j} \left[S_i(P)S_j(P) - \delta_{ij}S_i(P) + C_{ij}\right] \qquad \text{for all } i, j \tag{E3.12.36}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, and

$$S_i(P) = \frac{\partial Q(P)}{\partial P_i} \frac{P_i}{Q(P)}, \quad i = 1, \ldots, n. \tag{E3.12.37}$$

(E3.12.36) can be rewritten as

$$H_Q(P) = Q(P)\ \hat{P}^{-1} \left(C + S(P)S(P)' - \hat{S}(P)\right)\hat{P}^{-1}, \tag{E3.12.38}$$

where $\hat{P}$ is the diagonal matrix of P_is, $S(P)$ is the column vector of $S_i(P)$s and $\hat{S}(P)$ is the diagonal matrix formed by $S(P)$. In view of (E3.12.30) – (E3.12.33), we know that

37 Apart from the case in which $n = 2$, (E3.12.31) - (E3.12.34) are not sufficient to ensure that L is convex.

$$\Sigma_i S_i(P) \;=\; \Sigma_i B_i \;+\; \Sigma_i \Sigma_j C_{ij} lnP_j \;=\; 1.$$

Thus, for all $P \in L$, $0 \le S_i(P) \le 1$, $i = 1, \ldots, n$. It follows that for all $P \in L$, $S(P)S(P)' - \hat{S}(P)$ is negative semidefinite.[38] Hence, if C is negative semidefinite, then $H_Q(P)$ is negative semidefinite for all $P \in L$.

At this stage it may be objected that (E3.12.34) is overly restrictive. Perhaps the negative semidefiniteness of $H_Q(P)$ for all $P \in L$ could be guaranteed under a weaker prior restriction. However, this is not the case. Assume, for example, that we are able to find $P^* \in L$ so that

$$S_i(P^*) \;=\; B_i + \Sigma_j C_{ij} lnP_j^* \;=\; 1 \qquad \text{for } i = i^*$$

$$= 0 \qquad \text{for } i \neq i^*.$$

Then, we would have

$$S(P^*)S(P^*)' - \hat{S}(P^*) \;=\; 0$$

and $H_Q(P^*)$ would not be negative semidefinite if C were not negative semidefinite.

Nevertheless, it is clear that Jorgenson's concavity condition (E3.12.34) is more restricting on the parameter estimates than the corresponding condition, (E3.12.28), derived under the Taylor's-series interpretation of (E3.12.1). Jorgenson emphasizes the importance of his concavity condition in influencing his parameter estimates. It would be of interest, therefore, to repeat his work using condition (E3.12.28) in place of (E3.12.34).

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38 Let $Z(x) = x'(SS' - \hat{S})\,x$ where $S \ge 0$ and $\underline{1}'S=1$. We can write $Z(x)$ as

$$\begin{aligned}
Z(x) &= \Sigma_i\Sigma_j x_i x_j S_i S_j - \Sigma_i x_i^2 S_i \\
&= \Sigma_i \Sigma_{j\neq i} x_i x_j S_i S_j + \Sigma_i x_i^2 S_i^2 - \Sigma_i x_i^2 S_i \\
&= \Sigma_i \Sigma_{j\neq i} x_i x_j S_i S_j + \Sigma_i S_i (S_i - 1) x_i^2 \\
&= \Sigma_i \Sigma_{j\neq i} x_i x_j S_i S_j - \Sigma_i \Sigma_{j\neq i} S_i S_j x_i^2 \\
&= \Sigma_i \Sigma_{j\neq i} S_i S_j\,(x_i x_j - x_i^2) \\
&= \Sigma_i \Sigma_j S_i S_j\,(x_i x_j - x_i^2) \\
&= \tfrac{1}{2}\, \Sigma_i \Sigma_j S_i S_j\,(2x_i x_j - x_i^2 - x_j^2) \\
&= -\tfrac{1}{2}\, \Sigma_i \Sigma_j S_i S_j\,(x_i - x_j)^2 \;\le\; 0.
\end{aligned}$$

Exercise 3.13 Linearizing the demand functions for separable production and utility functions

Nearly all applied general equilibrium models use separable production functions and utility functions. Separability assumptions reduce the number of parameters requiring explicit evaluation. They also lead to simplifications in the representation of systems of demand equations. In this exercise, we give you various separable specifications for production and utility functions and ask you to derive demand equations in forms suitable for use in Johansen-style models.

We start with a definition of separability. A function $f(X_\alpha, X_\beta, \ldots,)$ is separable[39] with respect to the partition $N_1, \ldots, N_k$ if it can be written in the form

$$f(X_\alpha, X_\beta, \ldots) = V\big(g_1(X^{(1)}), g_2(X^{(2)}), \ldots, g_k(X^{(k)})\big), \qquad \text{(E3.13.1)}$$

where $N_1, \ldots, N_k$ are a non-overlapping coverage of the set $\{\alpha, \beta, \ldots\}$, and $X^{(j)}$ is the subvector of $(X_\alpha, X_\beta, \ldots)$ formed by the components X_η for which $\eta \in N_j$.

An example of a separable production function is the widely used specification

$$Y = \min\{CES(X_{(11)}, X_{(12)}), \ldots, CES(X_{(n+1,1)}, X_{(n+1,2)})\}, \qquad \text{(E3.13.2)}$$

where Y is output; $X_{(is)}$ for i = 1, ..., n, s = 1, 2, is the input of good i from source s with s = 1 referring to domestic products and s = 2 referring to imports; $X_{(n+1,s)}$ for s = 1,2 is the input of primary factor of type s with s = 1 indicating labor and s = 2 indicating capital; and the notation $CES(X_{(i1)}, X_{(i2)})$ means that $X_{(i1)}$ and $X_{(i2)}$ are to be combined according to a CES function (see (E3.9.1)). In this example, the V of (E3.13.1) is a Leontief function and the g_is are each CES functions of two inputs. Under (E3.13.2), output is viewed as a Leontief combination of effective inputs where effective inputs are CES combinations of domestic and imported materials[40] and of labor and capital. The underlying assumptions are that there is no substitution between different materials and between materials and primary factors. However, substitution can take place between domestic and imported materials of the same commodity classification and between labor and capital.

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39 The definition given here is for what is often called "weak separability" (see, for example, Katzner (1970, p. 28)).

40 This treatment of imports in general equilibrium modeling was pioneered by Armington (1969 and 1970).

(a) Assume that the production function has the form (E3.13.2). Assuming cost minimizing behavior, derive the percentage-change form for the input demand functions.

(b) Assume that the production function has the form

$$Y = CD\{CES(X_{(11)},X_{(12)}), \ldots, CES(X_{(n+1,1)},X_{(n+1,2)})\},$$

where CD denotes Cobb-Douglas function. Again derive the percentage-change form for the input demand functions.

(c) Letting Q be the number of households, assume that the consumption bundle, X_k/Q, $k = 1,\ldots, n$, of effective inputs for the average household is chosen to maximize the strictly quasiconcave utility function

$$U(X_1/Q, X_2/Q, \ldots, X_n/Q) \tag{E3.13.3}$$

subject to

$$X_k = CES(X_{(k1)},X_{(k2)}), \; k = 1, \ldots, n \tag{E3.13.4}$$

and

$$\sum_{k=1}^{n} \sum_{s=1}^{2} P_{(ks)}X_{(ks)} = M, \tag{E3.13.5}$$

where M is the aggregate household budget, and $X_{(is)}$ and $P_{(is)}$ are the quantity consumed and price of good i from source s, with s = 1 referring to domestic sources and s = 2 referring to imports. Show that the percentage-change form of the system of household demand equations may be written as

$$x_{(ks)} = x_k - \sigma_k\left(p_{(ks)} - \sum_{t=1}^{2} S_{(kt)}p_{(kt)}\right) \quad k = 1, \ldots, n, \; s = 1,2, \tag{E3.13.6}$$

with

$$x_k - q = \varepsilon_k(m - q) + \sum_{i=1}^{n} \eta_{ki}p_i, \quad k = 1, \ldots, n, \tag{E3.13.7}$$

and

$$p_k = \sum_{s=1}^{2} S_{(ks)}p_{(ks)}, \quad k = 1, \ldots, n, \tag{E3.13.8}$$

where $S_{(ks)}$ is the share of the household sector's expenditure on good k which is devoted to good k from source s, σ_k is the elasticity of substitution between the alternative types of good k, and the ε_ks and the η_{ki}s are expenditure and own- and cross-price elasticities satisfying the restrictions flowing from utility maximization, namely[41]

$$\sum_{k=1}^{n} \varepsilon_k \alpha_k = 1, \qquad \textit{(Engel's aggregation)} \quad \text{(E3.13.9)}$$

$$\sum_{i=1}^{n} \eta_{ki} = -\varepsilon_k,\ k = 1, \ldots, n, \qquad \textit{(homogeneity)} \quad \text{(E3.13.10)}$$

and

$$\alpha_i(\eta_{ik} + \varepsilon_i\alpha_k) = \alpha_k\,(\eta_{ki} + \varepsilon_k\alpha_i), \quad i \neq k, \quad \textit{(symmetry)} \quad \text{(E3.13.11)}$$

where α_k is the share of the household sector's budget devoted to good k from both sources.

Answer to Exercise 3.13

(a) Total costs, C, are given by

$$C = \sum_{i=1}^{n+1} \sum_{s=1}^{2} P_{(is)} X_{(is)}, \qquad \text{(E3.13.12)}$$

where $P_{(is)}$ is the cost of input i of type s. With the production function (E3.13.2), cost minimization requires that we put[42]

$$Y = \mathrm{CES}\big(X_{(k1)}, X_{(k2)}\big) \quad \text{for all } k = 1, \ldots, n+1. \qquad \text{(E3.13.13)}$$

Thus, to minimize (E3.13.12) subject to (E3.13.2), we must for each k choose $X_{(k1)}$ and $X_{(k2)}$ to minimize

$$\sum_{s=1}^{2} P_{(ks)} X_{(ks)} \qquad \text{(E3.13.14)}$$

subject to (E3.13.13). This later problem was studied in Exercise 3.9. By adapting (E3.9.12) to the notation of the present problem, we find that the percentage change form for the input demand system is

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41 For exercises on (E3.13.9) – (E3.13.11), see Dixon, Bowles and Kendrick (1980, Exercises 2.1, 2.2, 2.6 and 2.7).

42 If (E3.13.13) were not satisfied, we could cut costs while holding output constant by reducing the use of inputs k for which

$$Y < \mathrm{CES}(X_{(k1)}, X_{(k2)}).$$

$$x_{(ks)} = y - \sigma_k\left(p_{(ks)} - \sum_{t=1}^{2} S_{(kt)}p_{(kt)}\right), \quad k = 1, \ldots, n+1, \; s = 1,2, \tag{E3.13.15}$$

where the lower case symbols, $x_{(ks)}$, y and $p_{(ks)}$, are percentage changes in variables denoted by the corresponding upper case symbols, σ_k is the elasticity of substitution between the alternative types of input k and $S_{(ks)}$ is the share of input k of type s in the total cost of input k, i.e,

$$S_{(ks)} = P_{(ks)}X_{(ks)} / \Sigma_t P_{(kt)}X_{(kt)} \quad .$$

(b) Let

$$X_k = CES\left(X_{(k1)}, X_{(k2)}\right) \tag{E3.13.16}$$

i.e., let X_k denote the effective level of input k. If we are to minimize costs, then we must spend as little as possible on $X_{(k1)}$ and $X_{(k2)}$ subject to achieving the optimal level for X_k. Thus, $X_{(k1)}$ and $X_{(k2)}$ will minimize

$$\sum_{s=1}^{2} P_{(ks)}X_{(ks)} \tag{E3.13.17}$$

subject to (E3.13.16) with X_k set at its optimal level. It follows that movements in $X_{(ks)}$, $P_{(ks)}$, s = 1,2 and X_k will be related by

$$x_{(ks)} = x_k - \sigma_k\left(p_{(ks)} - \sum_{t=1}^{2} S_{(kt)}\, p_{(kt)}\right), \quad k = 1, \ldots, n+1, \; s = 1,2, \tag{E3.13.18}$$

where the notation is familiar from part (a).

To determine the x_ks we can consider the problem of choosing $X_1, X_2, \ldots, X_{n+1}$ to minimize

$$\sum_{k=1}^{n+1} P_k X_k \tag{E3.13.19}$$

subject to

$$Y = CD\left(X_1, \ldots, X_{n+1}\right) \;, \tag{E3.13.20}$$

where P_k is the minimum cost per unit of effective input k. From (E3.13.19) – (E3.13.20), we find that

$$x_k = y - \left(p_k - \sum_{i=1}^{n+1} \alpha_i p_i\right), \quad k = 1, \ldots, n+1, \tag{E3.13.21}$$

where α_i is the share of input i of both types in total costs. Next we note that P_k is given by

$$P_k = \left(\sum_{s=1}^{2} P_{(ks)} X_{(ks)} \right) / X_k \;, \qquad \text{(E3.13.22)}$$

where $X_{(k1)}$ and $X_{(k2)}$ minimize (E3.13.17) subject to (E3.13.16). From (E3.13.22) we obtain

$$p_k \;=\; \sum_{s=1}^{2} S_{(ks)} \left(p_{(ks)} + x_{(ks)} \right) - \; x_k \;. \qquad \text{(E3.13.23)}$$

On applying (E3.13.18) we find that

$$p_k \;=\; \sum_{s=1}^{2} S_{(ks)}\, p_{(ks)} \;. \qquad \text{(E3.13.24)}$$

Finally, we combine (E3.13.18), (E3.13.21) and (E3.13.24) to generate the percentage-change form for the input demand system as

$$x_{(ks)} \;=\; y \;-\; \sigma_k \left(p_{(ks)} - \sum_{t=1}^{2} S_{(kt)} p_{(kt)} \right)$$

$$- \left(\sum_{t=1}^{2} S_{(kt)} p_{(kt)} - \sum_{i=1}^{n+1} \sum_{t=1}^{2} \alpha_{(it)}\, p_{(it)} \right),$$

$$k = 1, \ldots, n+1, \; s = 1,2, \qquad \text{(E3.13.25)}$$

where $\alpha_{(is)}$ is the share of input i from source s in total costs, i.e. $\alpha_{(is)} = \alpha_i S_{(is)}$.

On the right hand side of (E3.13.25) we have two substitution terms, a within-group term and a between-group term. The first substitution term implies that if the price of input k of type s rises relative to the general price of input k, then the demand for input k of type s will fall relative to the overall demand for input k. The percentage change in the general price of input k is a cost-share weighted average of the percentage changes in $P_{(k1)}$ and $P_{(k2)}$. The second substitution term shows that if the general price of input k rises relative to a cost-share weighted index of all input prices, then there will be substitution away from both types of input k towards effective inputs of other materials or primary factors.

(c) If we are to maximize utility subject to budget constraint, then we must spend as little as possible in achieving whatever are the optimal levels for effective inputs. Hence, $X_{(k1)}$ and $X_{(k2)}$ will minimize

$$\sum_{s=1}^{2} P_{(ks)}X_{(ks)} \qquad \text{(E3.13.26)}$$

subject to

$$X_k = CES\big(X_{(k1)}, X_{(k2)}\big) \qquad \text{(E3.13.27)}$$

As we have noted earlier in this exercise, problem (E3.13.26) – (E3.13.27) leads to percentage-change equations of the form (E3.13.6).

To determine the effective input levels, we consider the problem of choosing $X_1, X_2, \ldots, X_n$ to maximize

$$U(X_1/Q, X_2/Q, \ldots, X_n/Q) \qquad \text{(E3.13.28)}$$

subject to

$$\sum_{k=1}^{n} P_kX_k = M, \qquad \text{(E3.13.29)}$$

where P_k is the minimum cost per unit of effective input k. We know from (E3.13.24) in part (b) that percentage movements in P_k are described by (E3.13.8). Thus, all that remains is to establish (E3.13.7).

We can rewrite problem (E3.13.28) – (E3.13.29) as follows: choose $X^*_1, \ldots, X^*_n$ to maximize

$$U(X^*_1, \ldots, X^*_n) \qquad \text{(E3.13.30)}$$

subject to

$$\sum_{k=1}^{n} P^*_kX^*_k = M, \qquad \text{(E3.13.31)}$$

where $X^*_k = X_k/Q$ and $P^*_k = P_kQ$. From (E3.13.30) – (E3.13.31) we have

$$x^*_k = \varepsilon_k m + \sum_{i=1}^{n} \eta_{ki} p^*_i, \quad k = 1, \ldots, n, \qquad \text{(E3.13.32)}$$

where the η_{ki}s and ε_ks satisfy the restrictions, (E3.13.9) – (E3.13.11), flowing from the utility maximizing model with α_k interpreted as

$$\alpha_k = P^*_kX^*_k/M = P_kX_k/M \quad \text{for all k.}$$

Since $x_k^* = x_k - q$ and $p_k^* = p_k + q$, (E3.13.32) implies that

$$x_k - q = \varepsilon_k m + \sum_{i=1}^{n} \eta_{ki}(p_i + q) . \tag{E3.13.33}$$

In view of (E3.13.10), (E3.13.33) can be rearranged as (E3.13.7).

The interpretation of the system (E3.13.6) – (E3.13.8) is straightforward. In (E3.13.6) we see that the demand for good k of type s moves with the demand for effective units of good k. However, if the price of good k of type s increases relative to the overall price of good k, then there will be substitution away from good k of type s towards the alternative source of good k. Movements in the overall price of good k are defined by (E3.13.8). Equation (E3.13.7) explains the movements in the demand per household for effective units of good k in terms of the movement in the budget per household and movements in the overall prices of all goods. The ε_k appearing in this equation is the expenditure elasticity for effective units of good k while the η_{ki}s are own- and cross- price elasticities for effective units of good k with respect to the overall prices, $P_1, \ldots, P_n$. Since the ε_ks and η_{ki}s satisfy the conditions flowing from utility maximization, in assigning values for them we are free to draw from the extensive literature on the systems approach to applied demand theory (see Powell (1974), Phlips (1974) and Deaton and Muellbauer (1980)).