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NOTES AND PROBLEMS IN MICROECONOMIC THEORY

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Chapter 2

THEORY OF THE CONSUMER: INTRODUCTION

2.1. Goals, reading guide and references

The theory of consumer demand is a basic building block for many economic studies. It plays a particularly important role in welfare economics, international trade theory, general equilibrium theory and the theory of public finance. In addition, the techniques required in consumer demand analysis are readily applicable to other parts of economic theory. You will find, for example, that the theory of production is very similar from a mathematical point of view to the theory of consumption.

The objective of this chapter is to introduce you to some of the main ideas of consumer theory and to give you a chance to practise the relevant techniques of analysis. We hope that by the time you have completed the readings and problems that you will have reached the following goals:

(1) a thorough understanding of the procedure for deriving demand functions from the model of consumer maximization of utility subject to a budget constraint;

(2) an ability to discuss clearly the meaning of the uniqueness of a utility function up to a monotonic transformation;

(3) an ability to establish the triad, i.e. the Engel aggregation, the homogeneity restriction, and the symmetry restriction;

(4) an understanding of the Hicks–Slutsky partition and an ability to prove that the own-price substitution effect is negative;

(5) an understanding of how preference orderings could be constructed using observed market behavior (revealed preference);

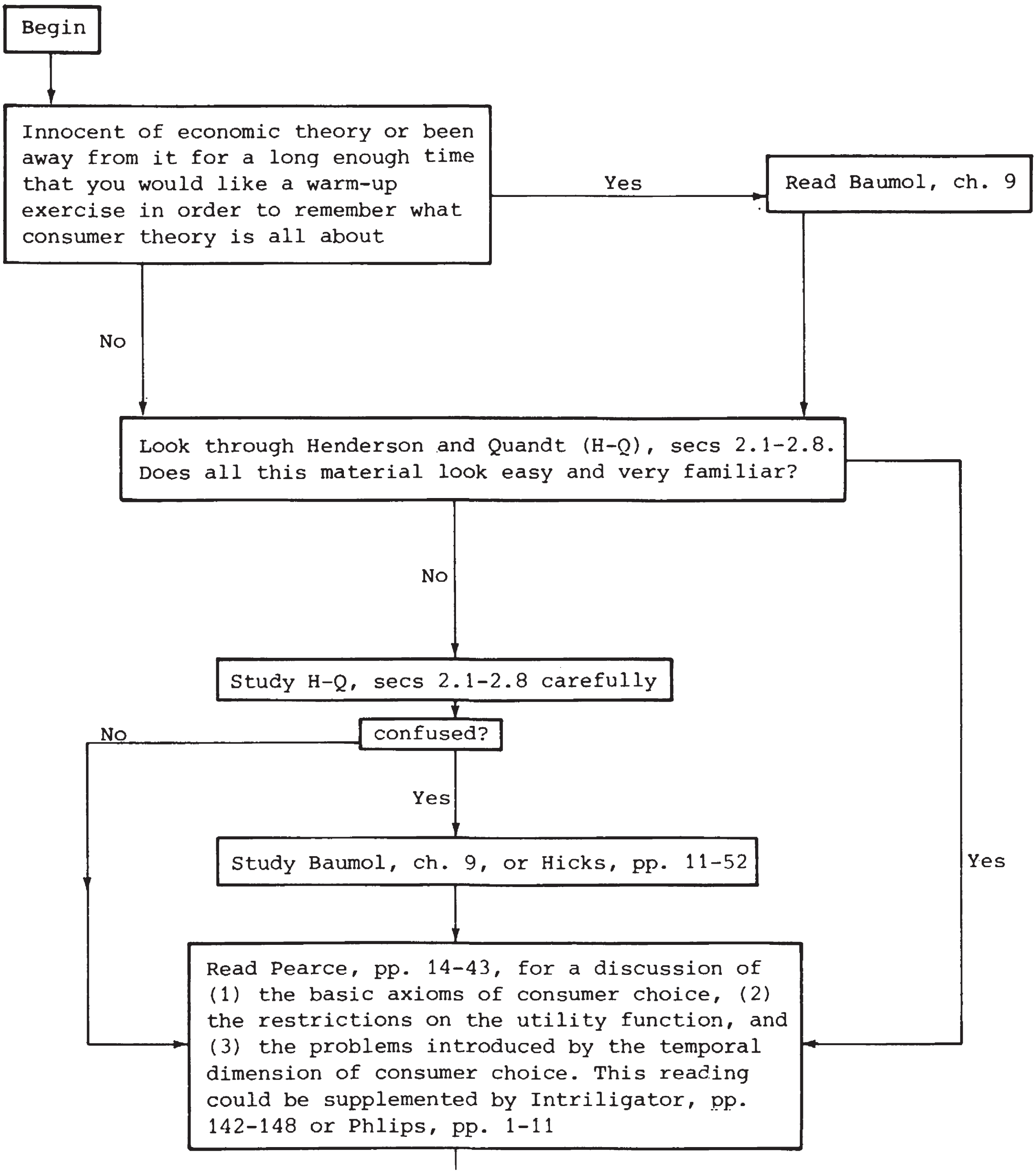
(6) a familiarity with the basic ideas of the pure theory of exchange, including some of the geometric tools used in general equilibrium analysis, e.g. offer curves, contract curves, and the Edgeworth box;

(7) an understanding of how consumer theory is used to provide restrictions on parameter values in the estimation of a complete system of commodity

demand functions. In particular, you should be familiar with the restrictions which flow from the adoption of an additive utility function.

Reading Guide 2 provides a suggested path through the readings to cover these concepts. Sections 2.2–2.4 are some short notes stating some of the principal definitions and theoretical results encountered in the problem set. Readings and references are given in abbreviated terms in the reading guide and in the rest of the chapter; full citations are in the reference list.

Reading Guide*



An important use of consumer theory is in the estimation of systems of demand equations. Here, the theory often allows the econometrician to place sufficient restrictions on parameter values to render estimation possible in the presence of data limitations. Read Powell (1974, chs. 1 and 2) and/or Theil (1967, pp. 182-200), or Theil (1975, pp. 1-30), or Phlips, pp. 32-66. Phlips contains some interesting problems which you might like to attempt.

This reading will be rather difficult for many of you. However, most of the main results of classical consumer theory and the recent work on the underlying theory for the estimation of consistent sets of demand equations are covered in the above references. If you can persevere, the reward is substantial.

Like to see an application of the additivity restrictions in the estimation of a complete set of demand equations

Yes

Read Powell (1966, pp. 661-675)

No

Want to obtain an impression of how complete systems of demand equations are used in economy-wide planning models

Yes

Skim Taylor and Black. Try to identify their treatment of consumer demand

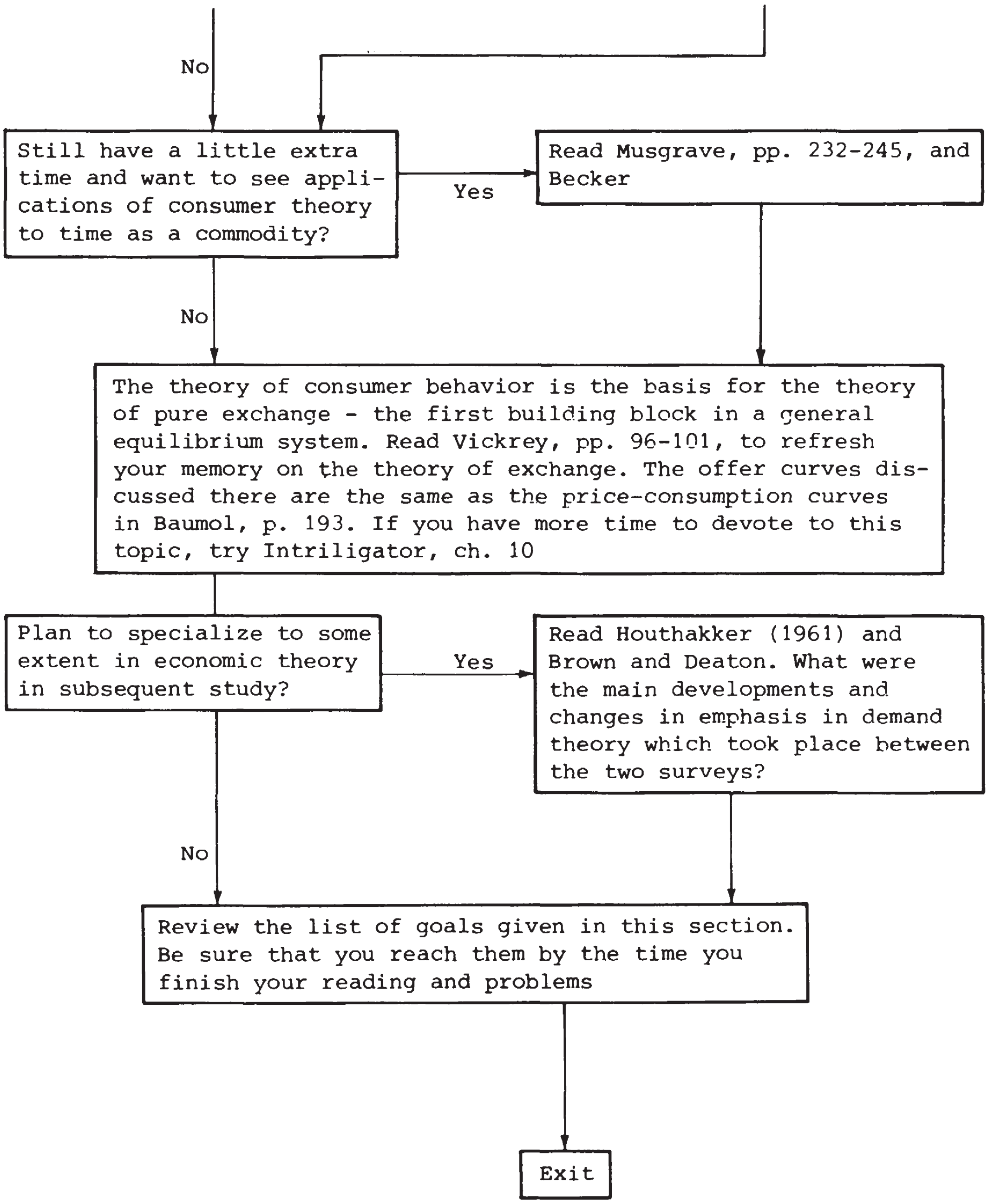
No

Read Pearce, pp. 64-72, and/or Little, pp. 283-289, and/or Intriligator, pp. 163-166, on the revealed preference approach

Running ahead of schedule and want to familiarize yourself with an alternative approach to the above material

Yes

Read Lancaster's (1966) reformation of consumer theory in terms of consumption activities rather than consumption goods and services



* For full citations, see reference list in this section.

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2.2. Notes on utility maximizing

Much theoretical and applied economics starts with the hypothesis that household preferences over alternative consumption bundles can be represented by a *utility function*. That is, we assume that a continuous function U can be defined so that

$$U(x) > U(y)$$

if and only if the household prefers commodity bundle x to commodity bundle y.[1]

There is no requirement, of course, that U be unique. In fact, if a household's preferences are representable by any function, then they are representable by many functions. If U is a valid utility function, then so is kU, where k is any positive number. In general, if f is a monotonically increasing[2] function, then $f(U)$ will provide a valid representation of the household's preferences. Notice that

$$f(U(x)) > f(U(y))$$

if and only if

$$U(x) > U(y),$$

i.e.

$$f(U(x)) > f(U(y))$$

if and only if the household prefers x to y.

On the other hand, not all sets of household preferences can be accommodated by a utility function. For example, if a household prefers x to y and y to z, but it also prefers z to x, then a utility function representation is ruled out. Because x is preferred to y, we require that U has the property that

$$U(x) > U(y). \tag{2.2.1}$$

Similarly, we require that

$$U(y) > U(z) \tag{2.2.2}$$

and

$$U(z) > U(x). \tag{2.2.3}$$

Clearly, it is impossible to define a function U which satisfies (2.2.1)–(2.2.3). Hence, by adopting a utility function representation, we are assuming that

[1] x and y are non-negative n-vectors whose components represent the household consumption levels for each commodity. We assume that U is defined for all $x \geqslant 0$, $x \in R^n$. Hence, we assume that commodities are divisible, i.e. any non-negative vector can be consumed.

[2] f is a monotonically increasing function if and only if

$$f(a) > f(b)$$

wherever

$$a > b.$$

household preferences are *transitive*, i.e. if the household prefers commodity bundle x to y and y to z, then the household will prefer x to z.

While transitivity is the most obvious restriction imposed by the utility function representation, there are others. If you want to know what these are, we suggest that you check either Phlips (1974, pp. 1–8) or Intriligator (1971, pp. 142–145). Both these readings give straightforward and complete statements of the conditions under which a set of preferences can be represented by a utility function. Our own view is that if we are happy to assume that preferences are transitive, then we should also be happy to accept the assumption that preferences are representable by a utility function. You will find when you read Phlips and Intriligator, that the assumptions (beyond transitivity) required to ensure the existence of a utility function have little practical significance.[3]

If we accept that it is reasonable to represent household preferences by a utility function, then the natural next step is to assume that households behave as if they choose their purchases of goods and services to maximize the utility derivable from their total budgets. We assume that the representative household chooses $x \geqslant 0$ to maximize

$$U(x) \tag{2.2.4}$$

subject to

$$p'x \leqslant y,$$

where x is an n-vector whose components are the amounts of each commodity purchased, p is the vector of commodity prices, y is the total household budget, and U is the household's utility function.

Many of you will have some familiarity with model (2.2.4), at least in its two-commodity form.[4] Where $n = 2$, we can illustrate the household utility-maximizing problem as in fig. 2.2.1. $\alpha\alpha$, $\beta\beta$ and $\gamma\gamma$ are contours (usually called indifference curves) of the utility function. $A'B$ is the budget line defining the household's feasible set of purchases and $\bar{x}$ is the problem solution, i.e. the commodity bundle which maximizes utility subject to the budget constraint. You will recall that the purpose of diagrams such as fig. 2.2.1 is to facilitate the discussion of how households are likely to react to changes in prices, p, and income, y.[5] Such changes generate shifts in the position of the budget line and many exercises in elementary courses are concerned with tracing out the implica-

[3] We return to this idea in Chapter 3, especially E3.4.

[4] Standard textbook treatments include Ferguson and Maurice (1974, chs. 3 and 4), Baumol (1972, ch. 9) and Hirshleifer (1976, chs. 3 and 4).

[5] y might be more appropriately referred to as the household's level of expenditure. In this book we follow conventional practice and call y 'income'.

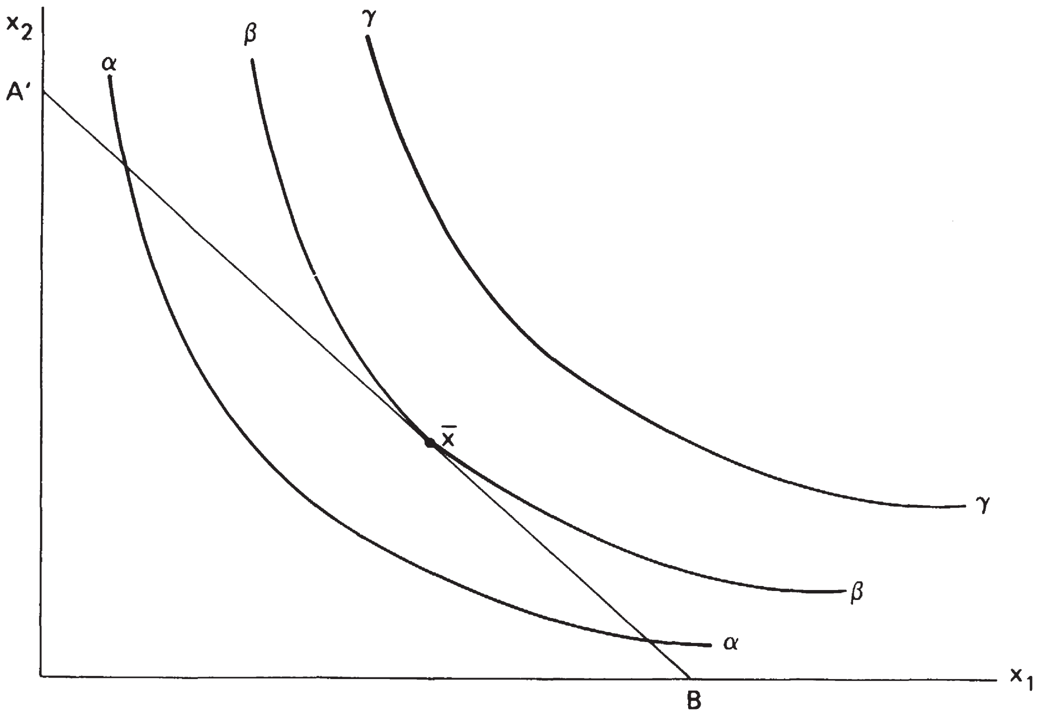

Figure 2.2.1

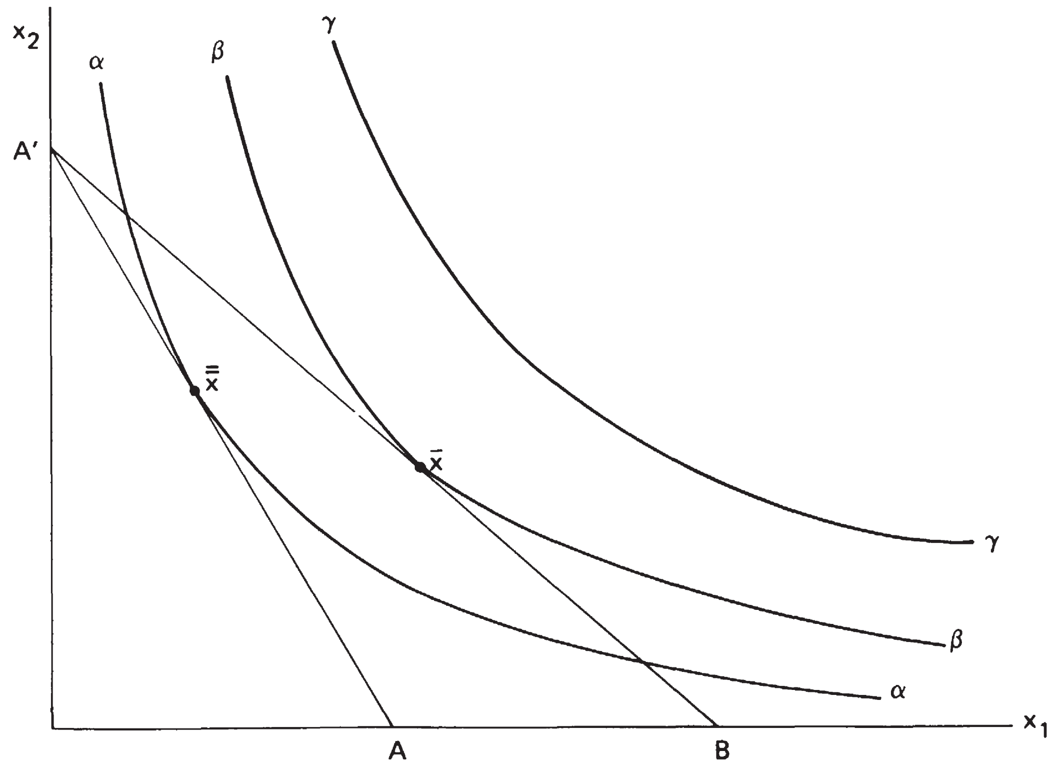

Figure 2.2.2. The increase in the price of good 1 shifts the budget line from $A'B$ to $A'A$. Household consumption moves from $\bar{x}$ to $\bar{\bar{x}}$.

tions for household consumption. For example, fig. 2.2.2 illustrates the impact on household purchases of goods 1 and 2 of an increase in p_1, with p_2 and y held constant. (Can you remember how to separate out the income and substitution effects? If not, look at Intriligator (1971, p. 161).) More generally, model (2.2.4) provides a theoretical structure for investigations of systems of household demand equations.

2.3. Systems of demand equations

On the basis of (2.2.4) we can write the system of household demand equations as

$$x_i = g_i(p_1, p_2, ..., p_n; y), \quad i = 1, ..., n, \tag{2.3.1}$$

where $g_i(p_1,...,p_n; y)$ denotes the solution for x_i in problem (2.2.4) when prices and income are at the levels $p_1,...,p_n$ and y. It is usual to assume that the g_i are functions, i.e. for each value of p and y the solution for (2.2.4) is unique. It is also usual to assume that the g_i are continuous. (Figs. 2.3.1 and 2.3.2 show counter-cases.) Both uniqueness and continuity are assured if the utility function is *strictly quasiconcave* (see E1.13). In this chapter we will not attempt a rigorous justification of the strict quasiconcavity assumption. We simply ask you to rely

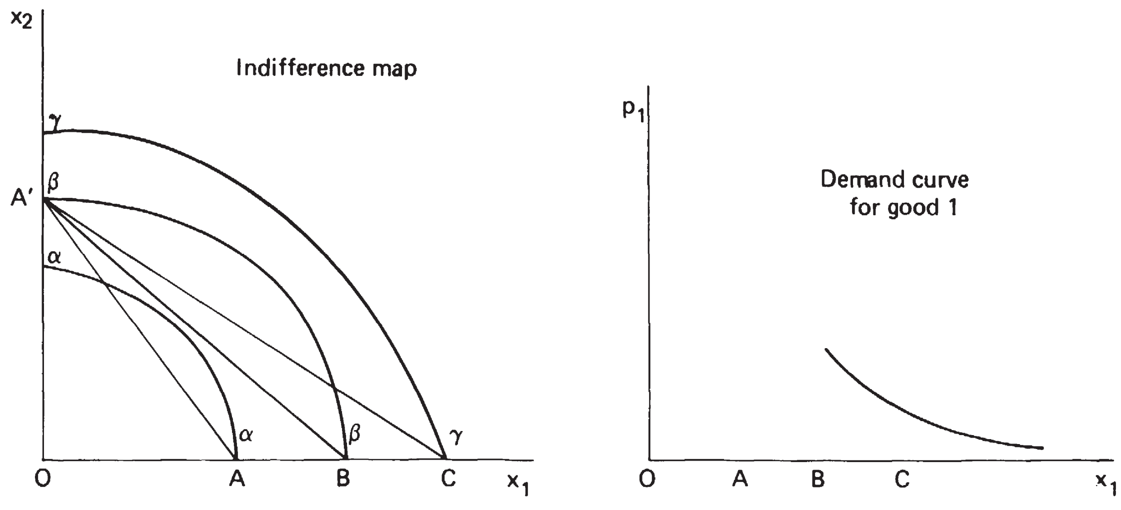

Figure 2.3.1. $\alpha\alpha$, $\beta\beta$ and $\gamma\gamma$ are indifference curves exhibiting the 'wrong' curvature. When the budget line is $A'A$, household consumption is completely specialized in good 2. (The solution to the consumer problem is at A'.) As we lower the price of good 1 (holding p_2 and y constant) consumption stays at A' until the budget line reaches $A'B$. At this stage the consumer is indifferent between points A' and B. As p_1 falls further, the consumer specializes in good 1. The resulting demand curve (illustrated on the right) exhibits a discontinuity at the price–income combination implied by budget line $A'B$.

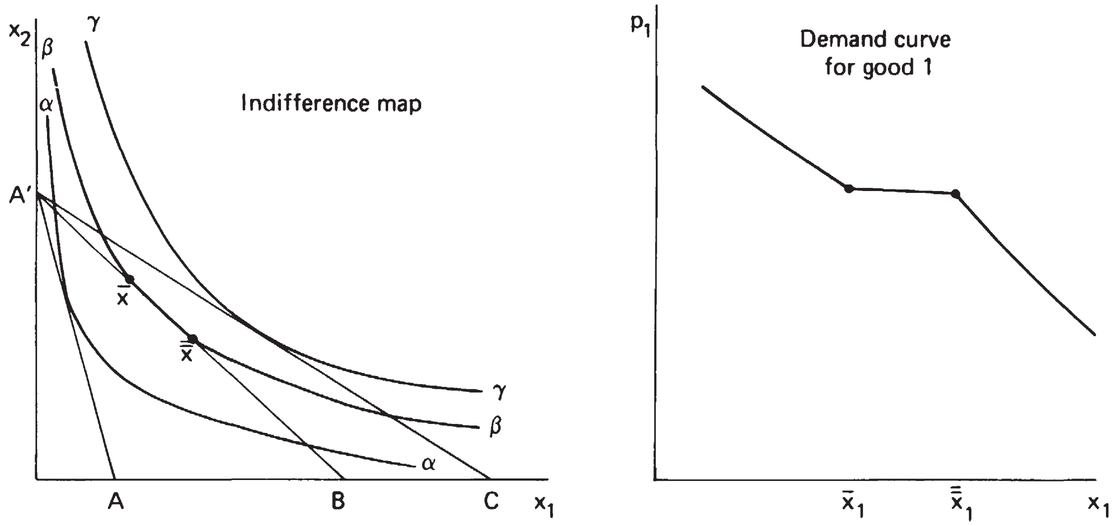

Figure 2.3.2. Again $\alpha\alpha$, $\beta\beta$ and $\gamma\gamma$ are indifference curves. This time, the indifference curve $\beta\beta$ has a linear segment. With the particular price–income combination implied by the budget line $A'B$, the household is indifferent over the range of consumption bundles from $\bar{x}$ to $\bar{\bar{x}}$, i.e. the consumption bundle is not uniquely determined. The demand curve for good 1 is illustrated on the right.

on the traditional story of diminishing marginal rates of substitution (see, for example, Ferguson and Maurice (1974, pp. 74–79), Hirshleifer (1976, pp. 68–69) and Baumol (1972, ch. 9)). In chapter 3 (see E3.4) we will be more ambitious. We will argue that if observable household behavior is consistent with any utility-maximizing model, then it is consistent with one in which the utility function is strictly quasiconcave. In other words, there is no set of data which would support the utility-maximizing model but at the same time allow us to reject the assumption that the utility function is strictly quasiconcave. Thus, if we are prepared to assume utility maximizing, then there is no reason to be shy about the additional assumption that the utility function is strictly quasiconcave.

A final assumption to be noted here is that the g_i are differentiable. This assumption does not really need to be justified independently of the continuity assumption. If the g_i are continuous, then there can be no harm in assuming them to be differentiable. If the 'truth' is that the g_i are continuous but not differentiable, then the truth can be approximated to any degree of precision by a model in which the g_i are assumed to be differentiable. The differentiability assumption is convenient because it allows us to define elasticities. We describe the responsiveness of the demand for good i to changes in the price of good j by

$$e_{ij} = \frac{\partial g_i}{\partial p_j} \frac{p_j}{g_i}, \quad \text{for all } i \text{ and } j, \tag{2.3.2}$$

where e_{ij} is called the elasticity of demand for good i with respect to price j. If $i = j$, then e_{ij} is an 'own' price elasticity. Otherwise it is a 'cross' price elasticity. e_{ij} gives the percentage rate of change in the demand for good i per percentage increase in p_j with all other prices and the expenditure level held constant. Similarly, we describe the responsiveness of demand to changes in income by income elasticities (E_i):

$$E_i = \frac{\partial g_i}{\partial y} \frac{y}{g_i}, \quad i = 1,\ldots n. \tag{2.3.3}$$

The focus of most applied work in demand systems is on the measurement of the elasticities e_{ij} and E_i.[6]

2.4. The implications of utility maximizing for demand systems

Most of the material in this chapter concentrates on the relationship between the model (2.2.4) and the system of demand equations (2.3.1). Some readers may, however, be wondering why we bother with (2.2.4). If our ultimate interest is in studying household demand, why do we not simply start with (2.3.1)? An objective of this chapter is to answer that question. Briefly, one role of model (2.2.4) is to suggest restrictions on the form of the functions g_i. For example, it can be shown that the model (2.2.4) implies the following three relationships between the elasticities e_{ij} and E_i:

$$\sum_k \alpha_k E_k = 1, \tag{2.4.1}$$

$$\sum_k e_{ik} = -E_i, \quad i = 1,\ldots,n, \tag{2.4.2}$$

$$\alpha_i(e_{ij} + E_i\alpha_j) = \alpha_j(e_{ji} + E_j\alpha_i), \qquad \text{for all } i \neq j, \tag{2.4.3}$$

where the E_k's and e_{ij}'s were defined by (2.3.2) and (2.3.3) and the α's are budget shares defined by

$$\alpha_i = p_i g_i / y,$$

i.e. α_i is the share of household expenditure devoted to good i. In E2.1, E2.2 and E2.6 you are asked to derive (2.4.1)–(2.4.3) from the model (2.2.4). For the present, however, we need only note the power of these restrictions. They provide $1 + n + (n^2-n)/2$ relationships between the elasticities. If n were 10, say,

[6] The elasticities are not normally assumed to be constants. Usual practice is to report elasticity estimates at sample mean values for p and y.

this would reduce the number of 'free' elasticities from 110 (n^2 price elasticities and n income elasticities) to 54. That is to say, given 54 of the e's and E's, we could compute the remaining 56 using (2.4.1)–(2.4.3) and observations of the budget shares.[7] Even greater economizing of free parameters in the demand system is possible if we are prepared to restrict the form of the utility function. In E2.17 you will find that if the utility function is additive, i.e.

$$U(x) = \sum_{i=1}^{n} U^i(x_i),$$

then (2.4.3) may be replaced by

$$e_{ij} = -E_i \alpha_j \left(1 + \frac{E_j}{\omega}\right), \quad \text{for all } i \neq j, \tag{2.4.4}$$

where ω is a scalar, independent of i and j. With (2.4.1), (2.4.2) and (2.4.4) we have $1 + n + (n^2 - n - 1) = n^2$ restrictions.[8] If $n = 10$, the number of free elasticities is reduced from 110 to 10. Since data on prices, income and consumption levels is always scarce, prior information such as (2.4.1)–(2.4.4) plays a critical role in econometric studies of demand systems. The smaller the number of free parameters to be estimated, the greater the chance that our limited data will yield statistically satisfactory results.[9]

A second role of model (2.2.4) is to give demand theory some normative as well as descriptive content. In many applications we will want to make descriptive statements, e.g. statements of the form 'household demand for clothing will increase by 2 percent in response to a 1 percent increase in income'. We may also want to make normative statements, e.g. 'a 10 percent increase in the price of food will reduce household welfare by the same amount as a 1 percent reduction in income'. Once we begin to make statements about consumer welfare, some form of welfare or utility function becomes essential. In this chapter we will touch on the role of model (2.2.4) in welfare economics. E2.16, for example, involves the concept of Pareto optimality. Our emphasis, however, will be on the first use of model (2.2.4), i.e. its role in econometrics. Applications to welfare economics will be taken up more fully in Chapter 3.

[7] Obviously not any 54 e's and E's would be adequate. For example, the 54 should not include all 10 of the E's.

[8] (2.4.4) consists of $n^2 - n$ equations, but it introduces one new parameter, ω. Thus, the number of restrictions is effectively $(n^2 - n - 1)$.

[9] For a short statement on this point see Phlips (1974, p. 56).

PROBLEM SET 2

Exercise 2.1. The Engel aggregation

Consider a consumer who chooses his consumption bundle, $x_1,\ldots,x_n$, to maximize his utility

$$U(x_1,\ldots,x_n) \tag{E2.1.1}$$

subject to his budget constraint

$$\sum_k p_k x_k \leqslant y. \tag{E2.1.2}$$

Prove the so-called Engel aggregation property, that the sum of the products of each income elasticity with its budget proportion must equal 1, i.e.

$$\sum_k \alpha_k E_k = 1, \tag{E2.1.3}$$

where

$$E_k = \frac{\partial x_k}{\partial y} \frac{y}{x_k}$$

and

$$\alpha_k = p_k x_k / y, \quad k = 1,\ldots,n.$$

Answer. We assume that the consumer spends his entire budget. (This follows from the usual assumption that marginal utilities are strictly positive.) Then by totally differentiating (E2.1.2) we find that

$$\sum_k (\mathrm{d}p_k) x_k + \sum_k (\mathrm{d}x_k) p_k = \mathrm{d}y.$$

If all prices are held constant and only income is varied, then

$$\sum_k (\mathrm{d}x_k) p_k = \mathrm{d}y.$$

Hence

$$\sum_k \frac{\partial x_k}{\partial y} p_k = 1. \tag{i}$$

Eq. (i) may be rewritten as

$$\sum_k \frac{\partial x_k}{\partial y} \frac{y}{x_k} \frac{x_k p_k}{y} = 1,$$

i.e.

$$\sum_k E_k \alpha_k = 1.$$

Exercise 2.2. Homogeneity restriction

Show that the demand equations of the form

$$x_i = \phi_i (p_1,\dots,p_n, y),$$

which can be derived from the model (E2.1.1)–(E2.1.2), are homogeneous of degree zero in prices and income. Now prove the 'homogeneity restriction', i.e.

$$\sum_k e_{ik} = -E_i, \quad i = 1,\dots,n, \tag{E2.2.1}$$

where

$$e_{ik} = \frac{\partial x_i}{\partial p_k} \frac{p_k}{x_i}$$

and

$$E_i = \frac{\partial x_i}{\partial y} \frac{y}{x_i}.$$

Answer. The demand functions are homogeneous of degree zero if

$$\phi_i(\beta p_1,\dots,\beta p_n, \beta y) = \phi_i(p_1,\dots,p_n, y) \tag{i}$$

for all $\beta > 0$. This means that if all prices and income are doubled, say, i.e. $\beta = 2$, then there is no change in commodity demands.

To show that the demand functions derived from the constrained utility maximizing model (E2.1.1)–(E2.1.2) are homogeneous of degree zero, we multiply the prices and income by a common factor $\beta > 0$. Then the constraint (E.2.1.2) becomes

$$\sum_k \beta p_k x_k \leqslant \beta y.$$

However, the constraint region is unaffected. The consumer's choice of consumption bundles is still restricted to the original set. Therefore, the utility-maximizing consumption bundle will remain unchanged. This is sufficient to justify (i).

(E2.2.1) may be derived as follows: by totally differentiating the demand functions we have

$$\mathrm{d}x_i = \sum_k \frac{\partial x_i}{\partial p_k}\,\mathrm{d}p_k + \frac{\partial x_i}{\partial y}\,\mathrm{d}y, \quad i = 1,\ldots,n.$$

These equations may be rewritten as

$$\frac{\mathrm{d}x_i}{x_i} = \sum_k \frac{\partial x_i}{\partial p_k}\,\frac{p_k}{x_i}\,\frac{\mathrm{d}p_k}{p_k} + \frac{\partial x_i}{\partial y}\,\frac{y}{x_i}\,\frac{\mathrm{d}y}{y},$$

i.e.

$$\frac{\mathrm{d}x_i}{x_i} = \sum_k e_{ik}\,\frac{\mathrm{d}p_k}{p_k} + E_i\,\frac{\mathrm{d}y}{y}, \qquad \text{for all } i.$$

If we set

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}p_k}{p_k} = \beta, \qquad \text{for all } k,$$

then we know from (i) that $\mathrm{d}x_i = 0$. Hence

$$0 = \sum_k e_{ik}\,\beta + E_i\beta,$$

i.e.

$$\sum_k e_{ik} = -E_i, \quad i = 1,\ldots,n.$$

Exercise 2.3. The linear expenditure system

(a) Here is a problem to test your facility at deriving the demand functions implied by a specific utility maximizing model. Assume that a consumer has an income of $\$y$ to divide between goods 1 and 2. What will be his demand for each good as a function of y, p_1 and p_2, where p_1 and p_2 are the commodity prices, if his utility function is

$$U(x_1, x_2) = \ln x_1 + 2 \ln x_2 \tag{E2.3.1}$$

and x_1 and x_2 are quantities of goods 1 and 2? (Check that the utility function is strictly quasiconcave over the positive orthant.)

(b) Answer the same question for another consumer whose utility function is

$$V(x_1, x_2) = x_1 x_2^2. \tag{E2.3.2}$$

Do you obtain the same results? Why? Can you be sure that (E2.3.2) is strictly quasiconcave?

(c) Now consider the more general case where

$$U(x_1, x_2) = \beta_1 \ln(x_1 - \gamma_1) + \beta_2 \ln(x_2 - \gamma_2) \tag{E2.3.3}$$

and $\beta_1, \beta_2, \gamma_1$ and γ_2 are parameters with $\beta_1, \beta_2 > 0$ and $\Sigma_i \beta_i = 1$.[10] Derive the demand functions for goods 1 and 2. Can you suggest why applied workers like to include the γ_i? *Hint*: what is the income elasticity of demand for each of the goods when the utility function has the form

$$U(x_1, x_2) = \beta_1 \ln x_1 + \beta_2 \ln x_2? \tag{E2.3.4}$$

Answer. (a) The first problem is to check that the utility function (E2.3.1) is strictly quasiconcave over the positive orthant. We consider any two positive consumption bundles $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Assume that both $U(x)$ and $U(y)$ are greater than or equal to γ. Then we will have shown that U is strictly quasiconcave over the positive orthant if we find that

$$U(\alpha x + (1-\alpha)y) > \gamma$$

for any α in the open interval (0, 1).

Under (E2.3.1),

$$U(\alpha x + (1-\alpha)y) = \ln(\alpha x_1 + (1-\alpha)y_1) + 2\ln(\alpha x_2 + (1-\alpha)y_2).$$

If $\alpha \in (0, 1)$, then

$$U(\alpha x + (1-\alpha)y) > \alpha \ln x_1 + (1-\alpha)\ln y_1 + \alpha\, 2\ln x_2 + (1-\alpha)\, 2\ln y_2.$$

(Draw a sketch to convince yourself that if $\alpha \in (0, 1)$ then

$$\ln(\alpha a + (1-\alpha)b) > \alpha \ln a + (1-\alpha)\ln b.)$$

Thus,

$$U(\alpha x + (1-\alpha)y) > \alpha\, U(x) + (1-\alpha)\, U(y) \geqslant \gamma.$$

This not only establishes that the utility function is *strictly* quasiconcave over the positive orthant, but also that it is *strictly* concave. U is said to be strictly concave if

$$U(\alpha x + (1-\alpha)y) > \alpha\, U(x) + (1-\alpha)\, U(y)$$

for all x and y, where $\alpha \in (0, 1)$.

[10] This is a convenient normalization. No loss of generality is implied. Why?

Turning now to the main problem, we note that the consumer will choose positive values for x_1 and x_2 to maximize

$$\ln x_1 + 2 \ln x_2$$

subject to

$$p_1 x_1 + p_2 x_2 \leqslant y.$$

The objective function is strictly quasiconcave over the positive orthant, the constraint set is convex and satisfies a constraint qualification. We can proceed with the Lagrangian method in confidence that it will yield the unique solution for the optimal values of x_1, x_2. On forming the Lagrangian

$$L(x, \lambda) = \ln x_1 + 2 \ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - y),$$

and equating the derivatives to zero, we find that at the consumption optimum

$$1/x_1 - \lambda p_1 = 0, \tag{E2.3.5}$$

$$2/x_2 - \lambda p_2 = 0 \tag{E2.3.6}$$

and

$$p_1 x_1 + p_2 x_2 = y. \tag{E2.3.7}$$

(We can ignore the sign restrictions on x_1 and x_2 and the possibility of the budget constraint being a strict inequality. Utility is $-\infty$ if either x_1 or x_2 is zero, and all the budget will be spent since $\partial U/\partial x_i > 0$ for $i = 1, 2$ and any values of the x_i.)

From (E2.3.5) and (E2.3.6) we find that

$$\frac{x_2}{2x_1} = \frac{p_1}{p_2}.$$

On substituting into (E2.3.7) we obtain

$$x_1 = y/3p_1 \tag{E2.3.8a}$$

and

$$x_2 = 2y/3p_2. \tag{E2.3.8b}$$

(E2.3.8) is the system of demand equations arising from the utility function (E2.3.1). Check that these demand equations are homogeneous of degree zero in prices and expenditure level.

(b) For the second consumer, the one having the utility function (E2.3.2), we notice that[11]

$$\ln (V(x_1, x_2)) = \ln x_1 + 2 \ln x_2 = U(x_1, x_2).$$

[11] It becomes tedious continually to mention that x_1 and x_2 are restricted to the positive orthant. For the remainder of this answer we will take that as understood.

Hence, ln V is exactly the same as the utility function, U, we had initially. Now ln V is a positive (or increasing) *monotonic transformation* of V. In general, $F(V)$ is said to be such a transformation if

$$\frac{\mathrm{d}F(V)}{\mathrm{d}V} > 0, \quad \text{for all values of } V.$$

This in turn implies

$$F(V_1) \gtreqless F(V_2)$$

if and only if $V_1 \gtreqless V_2$. In the particular case which we are studying in this problem, we have

$$\frac{\mathrm{d}F(V)}{\mathrm{d}V} = \frac{\mathrm{d}\ln V}{\mathrm{d}V} = \frac{1}{V} > 0.$$

Because U is a positive monotonic transformation of V, the demand systems derived from the two utility functions will be the same or, to put the same point another way, for given prices and incomes, the consumption pattern that maximizes a given utility function also maximizes all positive monotonic transformations of that function. See Henderson and Quandt (1971, sec. 2.3) if you need a fuller discussion.

Given that $F(V(x)) = U(x)$, that $U(x)$ is strictly quasiconcave, and that F is a positive monotonic transformation, can we be sure that $V(x)$ is strictly quasiconcave? Let x and y be any two consumption bundles such that

$$V(x) > \gamma \quad \text{and} \quad V(y) > \gamma.$$

Then because F is a monotonically increasing function, we can write

$$U(x) = F(V(x)) > F(\gamma)$$

and

$$U(y) = F(V(y)) > F(\gamma).$$

Now since U is strictly quasiconcave,

$$U(\alpha x + (1-\alpha)y) > F(\gamma)$$

for all $\alpha \in (0, 1)$. Therefore

$$F(V(\alpha x + (1-\alpha)y)) > F(\gamma)$$

and again, because F is monotonically increasing, we can conclude that

$$V(\alpha x + (1-\alpha)y) > \gamma.$$

This is sufficient to establish that V is strictly quasiconcave. Hence, we have shown that the property of strict quasiconcavity is preserved under positive monotonic transformations. On the other hand, the property of strict concavity is not necessarily preserved. Notice that while (E2.3.1) is strictly concave, (E2.3.2) is not.

(c) We form the Lagrangian

$$L(x,\lambda) = \beta_1 \ln(x_1-\gamma_1) + \beta_2 \ln(x_2-\gamma_2) - \lambda(p_1x_1 + p_2x_2 - y).$$

The first-order conditions are

$$\frac{\beta_1}{x_1-\gamma_1} = \lambda p_1, \tag{i}$$

$$\frac{\beta_2}{x_2-\gamma_2} = \lambda p_2 \tag{ii}$$

and

$$p_1x_1 + p_2x_2 = y. \tag{iii}$$

From (i) and (ii) we see that

$$\beta_i = \lambda(p_ix_i - p_i\gamma_i), \quad i = 1, 2.$$

Hence,

$$\sum_i \beta_i = 1 = \lambda\left(y - \sum_i p_i\gamma_i\right)$$

and

$$\lambda = 1\Big/\left(y - \sum_i p_i\gamma_i\right). \tag{iv}$$

Finally, we use (iv) in (i) and (ii) to obtain

$$x_k = \gamma_k + \frac{\beta_k\left(y - \Sigma_i p_i\gamma_i\right)}{p_k}, \quad k = 1, 2,$$

or alternatively

$$p_kx_k = p_k\gamma_k + \beta_k\left(y - \sum_i p_i\gamma_i\right), \quad k = 1, 2. \tag{v}$$

The equations (v) are the so-called *linear expenditure system* (LES). Expenditure p_kx_k on each good k is a *linear* function of prices and income. This explains much of the popularity in applied work of the n-commodity version of the utility function (E2.3.3); see for example Powell (1974, ch. 2) and the references given there. Notice that the γ_k play the key role of allowing expenditure elasticities of demand to differ from 1. If each of the γ_k is zero, i.e. the utility function is (E2.3.4), then

$$x_k = \frac{\beta_k y}{p_k}$$

and

$$E_k \equiv \frac{\partial x_k}{\partial y} \frac{y}{x_k} = 1.$$

A utility specification which forces expenditure elasticities to 1 is unsatisfactory in empirical applications. Expenditure elasticities are comparatively easily estimated. Typically, expenditure elasticities for 'food' are less than 1, whereas for 'recreation' they are more than 1. For a pioneering study of expenditure elasticities, see Houthakker (1957).

A common interpretation of the linear expenditure system (v) is as follows. γ_k is said to be the 'subsistence' requirement for good k. Then $\Sigma_i p_i \gamma_i$ is the subsistence level of expenditure and $(y - \Sigma_i p_i \gamma_i)$ is supernumerary expenditure, i.e. expenditure above subsistence requirements. β_k is the marginal budget share, i.e. β_k is the additional expenditure on good k associated with an additional dollar of total expenditure. Thus, system (v) is interpreted as meaning that expenditure on good k consists of two parts: a subsistence part, $p_k \gamma_k$, and a supernumerary part, $\beta_k(y - \Sigma_i p_i \gamma_i)$, with supernumerary expenditure on good k being proportional to total supernumerary expenditure. It should be pointed out, however, that this interpretation of the linear expenditure system frequently breaks down in empirical work. Often we find negative estimates for the γ_k's. Such results are incompatible with the idea that the γ_k's are subsistence quantities. As explained above, the key role of the γ's is to give system (v) flexibility with respect to the implied expenditure elasticities. Empirically and theoretically unjustified restrictions on the values of the expenditure elasticities can be introduced by attempts to restrict the γ's to positive values.

Exercise 2.4. The marginal utility of income

Return to the consumer described in the first part of E2.3. Assuming that the utility function is (E2.3.1), express the marginal utility of income as a function of p_1, p_2 and y. Repeat this exercise for the utility function (E2.3.2). Does the marginal utility of income change as we switch between the two utility functions? On the basis of your answer to that question, would you say that marginal utility is an ordinal or cardinal concept?

Answer. First we recall the analysis of Chapter 1, section 8. On the basis of that argument we know that the Lagrangian multiplier may be interpreted as the marginal utility of income, i.e.

$$\partial U/\partial y = \lambda,$$

where $\partial U/\partial y$ is the marginal utility of income, i.e. the rate of increase in utility per unit increase in y, and λ is the Lagrangian multiplier associated with the solution to the utility maximizing problem. Then from (E2.3.5) and (E2.3.6) we find that

$$\lambda p_1 x_1 = 1$$

and

$$\lambda p_2 x_2 = 2.$$

Hence

$$\lambda y = 3$$

and

$$\lambda = 3/y. \tag{i}$$

With the utility function (E2.3.2), the first-order conditions are

$$x_2^2 - \lambda^* p_1 = 0,$$

$$2x_1 x_2 - \lambda^* p_2 = 0$$

and

$$p_1 x_1 + p_2 x_2 = y.$$

These give

$$\left.\begin{aligned} &\lambda^* p_1 x_1 = x_1 x_2^2 \\ &\text{and} \\ &\lambda^* p_2 x_2 = 2x_1 x_2^2. \end{aligned}\right\} \tag{ii}$$

Hence

$$p_2 x_2 = 2p_1 x_1,$$

$$x_1 = y/3p_1$$

and

$$x_2 = 2y/3p_2.$$

Substituting back into (ii), we obtain

$$\lambda^* = \frac{4y^2}{9p_2^2 p_1}. \tag{iii}$$

By comparing (iii) and (i) we see that although a positive monotonic transformation of the utility function leaves the demand system unchanged, it *does* change the marginal utility of income. Thus, we may conclude that marginal utility is a *cardinal* concept. It depends on which particular utility function is selected from the family of equivalent utility functions describing the consumer's given set of preferences.

Exercise 2.5. Displacement analysis, an example

Consider the consumer described in the first part of E2.3. Adopting the utility function (E2.3.1), but without deriving the demand functions, find equations suitable for evaluating the changes in x_1, x_2 and λ, the marginal utility of income, which will result from small changes in the prices p_1 and p_2 and in income y.[12] From these equations, determine the income effects $\partial x_1/\partial y$, $\partial x_2/\partial y$, and the effects of changes in p_1 on x_1 and x_2, i.e. $\partial x_1/\partial p_1$ and $\partial x_2/\partial p_1$. What are the signs of the income effects? What property of the utility function is responsible for this result? Show that the cross substitution effect, $\partial x_2/\partial p_1$, is zero. You may check your calculations of the various partial derivatives by making appropriate differentiations of the demand functions which you derived in E2.3.

Answer. We assume that when prices and income change, the consumer reorganizes his purchases so that he maximizes his utility subject to his new budget constraint. At his new consumption levels, $x_1(N)$, $x_2(N)$, there will exist $\lambda(N)$ such that

$$\frac{1}{x_1(N)} - \lambda(N)\, p_1(N) = 0, \tag{i}$$

$$\frac{2}{x_2(N)} - \lambda(N)\, p_2(N) = 0 \tag{ii}$$

and

$$p_1(N)\, x_1(N) + p_2(N)\, x_2(N) = y(N), \tag{iii}$$

where $p_1(N)$, $p_2(N)$ and $y(N)$ are the new levels for p_1, p_2 and y.

Next, we compare the original first-order conditions (E2.3.5)–(E2.3.7) with the new first-order conditions (i)–(iii). By subtracting (E2.3.5) from (i) we obtain

[12] If you have difficulty with this problem, you should review the relevant theory in Intriligator (1971, ch. 7, esp. sec. 7.4).

$$\left(\frac{1}{x_1(N)} - \frac{1}{x_1}\right) - (\lambda(N)\, p_1(N) - \lambda p_1) = 0$$

i.e.

$$d(1/x_1) - d(\lambda p_1) = 0,$$

where $d(1/x_1)$ and $d(\lambda_1 p_1)$ are the changes in $1/x_1$ and λp_1. If the changes are small, this last equation becomes

$$\frac{\partial(1/x_1)}{\partial x_1}\, dx_1 - \frac{\partial(\lambda p_1)}{\partial \lambda}\, d\lambda - \frac{\partial(\lambda p_1)}{\partial p_1}\, dp_1 = 0,$$

i.e.

$$-\frac{1}{x_1^2}\, dx_1 - p_1\, d\lambda - \lambda dp_1 = 0. \tag{iv}$$

Similarly, by totally differentiating (E2.3.6) and (E2.3.7) we find that

$$-\frac{2}{x_2^2}\, dx_2 - p_2 d\lambda - \lambda dp_2 = 0 \tag{v}$$

and

$$p_1 dx_1 + p_2 dx_2 + (dp_1)x_1 + (dp_2)x_2 = dy. \tag{vi}$$

The three equations (iv)–(vi) may be set out in a convenient matrix format as

$$\begin{bmatrix} -\frac{1}{x_1^2} & 0 & p_1 \\ 0 & -\frac{2}{x_2^2} & p_2 \\ p_1 & p_2 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ dy - \sum_i (dp_i)x_i \end{bmatrix}. \tag{vii}$$

We 'solve' (vii) by writing

$$\begin{bmatrix} dx_1 \\ dx_2 \\ -d\lambda \end{bmatrix} = \begin{bmatrix} -\frac{1}{x_1^2} & 0 & p_1 \\ 0 & -\frac{2}{x_2^2} & p_2 \\ p_1 & p_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ dy - \sum_i (dp_i)x_i \end{bmatrix}.$$

The matrix to be inverted is the so-called bordered Hessian of the utility function (E2.3.1). On carrying out the inversion we find that

$$\begin{bmatrix} dx_1 \\ dx_2 \\ -d\lambda \end{bmatrix} = \frac{1}{\left(\frac{p_2}{x_1}\right)^2 + 2\left(\frac{p_1}{x_2}\right)^2} \begin{bmatrix} -p_2^2 & p_1 p_2 & \frac{2p_1}{x_2^2} \\ p_1 p_2 & -p_1^2 & \frac{p_2}{x_1^2} \\ \frac{2p_1}{x_2^2} & \frac{p_2}{x_1^2} & \frac{2}{x_1^2 x_2^2} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ dy - \sum_i (dp_i)x_i \end{bmatrix}. \tag{viii}$$

(If you have trouble with inverting matrices, check that we have the right answer by multiplying our inverse by the original bordered Hessian. You should generate the identity matrix.)

Eq. (viii) is suitable for evaluating the effects of changes in prices and income on consumption levels and on the marginal utility of income. We have derived it without explicitly obtaining the demand system (E2.3.8). In the present example this is not very useful. However, the technique becomes important when the demand functions cannot be derived explicitly. The point is illustrated in E2.6 and E.2.17.

On the basis of eq. (viii), we can compute the income derivatives, $\partial x_1/\partial y$ and $\partial x_2/\partial y$, as follows: if we consider a situation in which prices are fixed, but income changes, then

$$\begin{bmatrix} dx_1 \\ dx_2 \\ -d\lambda \end{bmatrix} = \frac{1}{\left(\frac{p_2}{x_1}\right)^2 + 2\left(\frac{p_1}{x_2}\right)^2} \begin{bmatrix} \frac{2p_1}{x_2^2} \\ \frac{p_2}{x_1^2} \\ \frac{2}{x_1^2 x_2^2} \end{bmatrix} dy.$$

Hence,

$$\frac{\partial x_1}{\partial y} = \frac{2p_1/x_2^2}{(p_2/x_1)^2 + 2(p_1/x_2)^2} \tag{ix}$$

and

$$\frac{\partial x_2}{\partial y} = \frac{p_2/x_1^2}{(p_2/x_1)^2 + 2(p_1/x_2)^2}$$

Both income derivatives are positive. This result is attributable to the additivity of the utility function. With this type of utility function, quantities consumed of any good do not affect the marginal utility of other goods. Hence, increases in income must result in increases in the purchases of all goods, for otherwise the marginal utility of a dollar's worth of expenditure on some goods would be left higher than that on others. Accordingly, the additive utility function excludes the possibility of inferior goods.

To determine the price derivatives $\partial x_1/\partial p_1$ and $\partial x_2/\partial p_1$, we set $\mathrm{d}y$ and $\mathrm{d}p_2$ equal to zero. Then (viii) implies that

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{bmatrix} = \frac{1}{(p_2/x_1)^2 + 2(p_1/x_2)^2} \begin{bmatrix} -p_2^2 & 2p_1/x_2^2 \\ p_1p_2 & p_2/x_1^2 \end{bmatrix} \begin{bmatrix} \lambda \mathrm{d}p_1 \\ -x_1 \mathrm{d}p_1 \end{bmatrix}.$$

From here we find that

$$\frac{\partial x_1}{\partial p_1} = \frac{1}{(p_2/x_1)^2 + 2(p_1/x_2)^2}\left(-\lambda p_2^2 - \frac{2x_1p_1}{x_2^2}\right)$$

and

$$\frac{\partial x_2}{\partial p_1} = \frac{1}{(p_2/x_1)^2 + 2(p_1/x_2)^2}\left(\lambda p_1p_2 - \frac{p_2}{x_1}\right).$$

Finally, if we use the first-order condition (E2.3.5), we see that

$$\lambda = 1/p_1x_1$$

so that the cross substitution effect, $\partial x_2/\partial p_1$, is zero.

The derivations of $\partial x_1/\partial y$, $\partial x_2/\partial y$, etc. can be checked by differentiating in the demand system (E2.3.8). For example, we find that

$$\frac{\partial x_2}{\partial p_1} = 0$$

and

$$\frac{\partial x_1}{\partial y} = \frac{1}{3p_1}.$$

(This last result may be obtained from (ix) by substituting from (E2.3.5) and (E2.3.6) to eliminate the x's and then simplifying.)

Exercise 2.6. Displacement analysis and the symmetry restriction

In E2.5, working with a special case, you obtained an expression for evaluating changes in the consumption of various commodities as a function of changes in prices and income. Now consider the general case where there are n goods $x_1,\ldots,x_n$; their prices are $p_1,\ldots,p_n$; and the consumer has income y and a strictly quasiconcave utility function $U(x_1,\ldots,x_n)$. Show that

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ -\mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} \lambda \mathrm{d}p \\ \mathrm{d}y - (\mathrm{d}p)'x \end{bmatrix}, \tag{E2.6.1}$$

where H is the Hessian matrix of the utility function, i.e.

$$H \equiv [U_{ij}]_{n \times n}, \quad \text{with } U_{ij} \equiv \frac{\partial^2 U}{\partial x_i \, \partial x_j},$$

p and x are vectors of prices and consumptions, and λ is the marginal utility of income.

Adopt the notation

$$R \equiv \begin{bmatrix} H & p \\ p' & 0 \end{bmatrix}^{-1} \equiv \begin{bmatrix} r_{11}, & \cdots & r_{1n}, & r_{1\,n+1} \\ \vdots & & & \\ r_{n1}, & \cdots & r_{nn}, & r_{n\,n+1} \\ & & & \\ r_{n+1\,1}, \cdots & & & r_{n+1\,n+1} \end{bmatrix}.$$

Note that R is a symmetric matrix. (Why?) Now prove the symmetry restriction, i.e.

$$\alpha_i(e_{ij} + E_i\alpha_j) = \alpha_j(e_{ji} + E_j\alpha_i), \quad \text{for all } i \neq j, \tag{E2.6.2}$$

where the α's, E's and e's are defined as in E2.1 and E2.2.

Answer. We assume that the consumer chooses $x_1, x_2,\ldots,x_n$ to maximize

$$U(x_1,\ldots,x_n)$$

subject to

$$\sum_k p_k x_k = y.$$

At a constrained maximum there will exist λ, which may be interpreted as the marginal utility of income, such that $\lambda, x_1, \ldots, x_n$, jointly satisfy

$$\frac{\partial U}{\partial x_i} = \lambda p_i, \quad i = 1, \ldots, n \tag{i}$$

and

$$\sum_k p_k x_k = y. \tag{ii}$$

By totally differentiating (i) and (ii) we find that

$$\sum_k \frac{\partial}{\partial x_k}\left(\frac{\partial U}{\partial x_i}\right) \mathrm{d}x_k = (\mathrm{d}\lambda)p_i + \lambda(\mathrm{d}p_i), \quad i = 1, \ldots, n,$$

and

$$\sum_k (\mathrm{d}p_k)\, x_k + \sum_k p_k(\mathrm{d}x_k) = \mathrm{d}y.$$

In matrix notation, these equations may be presented as

$$\begin{bmatrix} U_{11}, & \cdots & U_{1n}, & p_1 \\ & & & \\ U_{n1}, & \cdots & U_{nn}, & p_n \\ & & & \\ p_1, & \cdots & p_n, & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \\ \\ -\mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} \lambda \mathrm{d}p_1 \\ \vdots \\ \lambda \mathrm{d}p_n \\ \\ \mathrm{d}y - \sum_k (\mathrm{d}p_k)x_k \end{bmatrix}, \tag{iii}$$

i.e.

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ -\mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} \lambda \mathrm{d}p \\ \mathrm{d}y - (\mathrm{d}p)'x \end{bmatrix}.$$

The bordered Hessian,

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix},$$

is symmetric (from Young's theorem[13] $U_{ij} = U_{ji}$ for all i and j). Thus, R, the inverse of the bordered Hessian, is also symmetric.[14] (If A is any symmetric matrix, then A^{-1}, if it exists, is symmetric.)[15]

On solving (iii) we obtain

$$\begin{bmatrix} \mathrm{d}x \\ \\ -\mathrm{d}\lambda \end{bmatrix} = R \begin{bmatrix} \lambda \mathrm{d}p \\ \\ \mathrm{d}y - (\mathrm{d}p)'x \end{bmatrix}$$

or, more fully,

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \vdots \\ \mathrm{d}x_n \\ -\mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} r_{11}, & \dots r_{1n}, & r_{1n+1} \\ & & \\ r_{n1}, & \dots r_{nn}, & r_{nn+1} \\ r_{n+11} & \dots & r_{n+1\,n+1} \end{bmatrix} \begin{bmatrix} \lambda \mathrm{d}p_1 \\ \vdots \\ \lambda \mathrm{d}p_n \\ \mathrm{d}y - \sum_k (\mathrm{d}p_k)x_k \end{bmatrix} \tag{iv}$$

From (iv) we can deduce the derivatives of the demand functions. In particular

$$\frac{\partial x_k}{\partial y} = r_{kn+1}, \quad k = 1,\dots,n \tag{v}$$

(set $\mathrm{d}y = 1$, and $\mathrm{d}p_i = 0$, $i = 1,\dots,n$) and

$$\frac{\partial x_k}{\partial p_s} = \lambda r_{ks} - (r_{kn+1})x_s, \quad \text{for all } k \text{ and } s \tag{vi}$$

(set $\mathrm{d}p_s = 1$, and all the other price changes and $\mathrm{d}y$ to zero). Because $r_{ij} = r_{ji}$, eq. (vi) implies that

$$\frac{\partial x_i}{\partial p_j} - \frac{\partial x_j}{\partial p_i} = -(r_{in+1})x_j + (r_{jn+1})x_i. \tag{vii}$$

[13] See, for example, Hilton (1960, pp. 49–51).

[14] The strict quasiconcavity of U ensures that the consumer's utility-maximizing problem has one, and only one, solution for each choice of $p > 0$ and $y > 0$. This in turn ensures the existence of R. However, we usually make the additional assumption that H^{-1} exists. This is convenient, but it does not follow from the assumption of strict quasiconcavity (or even strict concavity). For example, consider the problem of choosing the scalar, x, to maximize the strictly quasiconcave objective function $(x\text{-}1)^3$ subject to $x \leqslant 1$. You will find that at the optimum value for x, ($x = 1$), the Hessian is zero, but that the R matrix exists.

[15] $(A^{-1})'A' = (AA^{-1})' = I$. Hence, $(A')^{-1} = (A^{-1})'$. If A is symmetric then it follows that $A^{-1} = (A^{-1})'$. Hence, A^{-1} is symmetric.

Then from (v) and (vii) we obtain

$$\frac{\partial x_i}{\partial p_j} - \frac{\partial x_j}{\partial p_i} = -\frac{\partial x_i}{\partial y} x_j + \frac{\partial x_j}{\partial y} x_i. \qquad \text{(viii)}$$

The final step in the derivation of the symmetry restriction is to translate the derivatives into elasticities. In (ix) the terms of the original equation (viii) are displayed in square brackets. The translation to elasticities is made in the usual way by multiplying and dividing through by prices, income and quantities:

$$\frac{x_i p_i}{y} \frac{p_j}{x_i} \left[\frac{\partial x_i}{\partial p_j}\right] \frac{y}{p_j p_i} - \frac{x_j p_j}{y} \frac{p_i}{x_j} \left[\frac{\partial x_j}{\partial p_i}\right] \frac{y}{p_i p_j}$$

$$= -\frac{x_i p_i}{y} \frac{y}{x_i} \left[\frac{\partial x_i}{\partial y}\right] \frac{[x_j] p_j}{y} \frac{y}{p_i p_j} + \frac{x_j p_j}{y} \frac{y}{x_j} \left[\frac{\partial x_j}{\partial y}\right] \frac{[x_i] p_i}{y} \frac{y}{p_i p_j}, \qquad \text{(ix)}$$

i.e.

$$\alpha_i e_{ij} - \alpha_j e_{ji} = -\alpha_i E_i \alpha_j + \alpha_j E_j \alpha_i,$$

and this last equation can be rearranged to give (E2.6.2).

Exercise 2.7. The triad

Alan Powell (1974) has referred to the three results, the Engel aggregation (E2.1.3), the homogeneity restriction (E2.2.1), and the symmetry restriction (E2.6.2), as the 'triad'. Some appreciation of the power of the triad can be gained by completing the following scheme:[16]

$\alpha_1 = \frac{1}{2}$,	$\alpha_2 = \frac{1}{4}$,	$\alpha_3 = \frac{1}{4}$	
$e_{11} = -1$,	$e_{12} = ?$,	$e_{13} = 0$	$E_1 = \frac{1}{2}$
$e_{21} = ?$,	$e_{22} = ?$,	$e_{23} = ?$	$E_2 = ?$
$e_{31} = ?$,	$e_{32} = -\frac{1}{4}$,	$e_{33} = ?$	$E_3 = 1$

[16] The E's, α's and e's are defined as in E2.1 and E2.2.

Answer. The completed table is shown below:

$\alpha_1 = \frac{1}{2}$,	$\alpha_2 = \frac{1}{4}$,	$\alpha_3 = \frac{1}{4}$	
$e_{11} = -1$,	$e_{12} = \frac{1}{2}$,	$e_{13} = 0$	$E_1 = \frac{1}{2}$
$e_{21} = \frac{1}{4}$,	$e_{22} = -1\frac{3}{4}$,	$e_{23} = -\frac{1}{2}$	$E_2 = 2$
$e_{31} = -\frac{1}{4}$,	$e_{32} = -\frac{1}{4}$,	$e_{33} = -\frac{1}{2}$	$E_3 = 1$

Starting from the incomplete table, we used the homegeneity restriction to find e_{12}, i.e.

$$e_{12} = -E_1 - e_{11} - e_{13}.$$

E_2 was computed from the Engel aggregation, i.e.

$$E_2 = (1 - E_1\alpha_1 - E_3\alpha_3)/\alpha_2,$$

The symmetry restriction allowed us to compute e_{21} from knowing e_{12}, the E's and the α's:

$$e_{21} = \frac{\alpha_1(e_{12} + E_1\alpha_2)}{\alpha_2} - E_2\alpha_1.$$

Similarly, we used the symmetry restriction to generate e_{31} and e_{23}. Finally, e_{22} and e_{33} were deduced from the homogeneity restriction.

On the basis of knowing the budget shares, the α_i's, and five of the elasticities, we were able to deduce the other seven elasticities. The triad provides $1 + n + n(n\text{-}1)/2$ restrictions. When $n = 3$ the triad suggests seven independent relationships between the elasticities.

The estimation of demand elasticities, E_i and e_{ij}, has been the objective of intensive econometric activity. Excellent survey texts are available, for example, Powell (1974) and Phlips (1974). Both these references emphasize the importance of prior restrictions in econometric work. It is too much to ask 'the data' to reveal $n^2 + n$ elasticities without some help from economic theory. The imposition of the triad restrictions, and perhaps some additional restrictions (see E2.17), has made possible the estimation of complete systems of demand elasticities.

Exercise 2.8. The Cournot aggregation

Prove that

$$\sum_i \alpha_i e_{ik} = -\alpha_k \qquad \text{for all } k.$$

Is this a further restriction on the demand elasticities, or is it implied by the triad?

Answer. We assume that

$$\sum_i p_i x_i = y.$$

By total differentiation we obtain

$$\sum_i (\mathrm{d}p_i) x_i + \sum_i (\mathrm{d}x_i) p_i = \mathrm{d}y.$$

Now we set all the price and income changes to zero with the exception of $\mathrm{d}p_k$. Thus,

$$(\mathrm{d}p_k) x_k + \sum_i (\mathrm{d}x_i) p_i = 0$$

and (i)

$$-x_k = \sum_i \frac{\partial x_i}{\partial p_k} p_i.$$

Finally, we translate into elasticities and budget shares by rewriting (i) as

$$\frac{[-x_k]p_k}{y} = \sum_i \left[\frac{\partial x_i}{\partial p_k}\right] \frac{p_k}{x_i} \frac{x_i[p_i]}{y},$$

i.e.

$$-\alpha_k = \sum_i e_{ik}\alpha_i. \tag{ii}$$

Restriction (ii) cannot be counted as additional to the triad. It can be deduced as follows: the symmetry restrictions imply

$$\begin{aligned}\sum_i e_{ik}\alpha_i &= \sum_i [\alpha_k(e_{ki} + E_k\alpha_i) - E_i\alpha_i\alpha_k] \\ &= \alpha_k \sum_i e_{ki} + \alpha_k E_k \sum_i \alpha_i - \alpha_k \sum_i \alpha_i E_i \\ &= -\alpha_k E_k + \alpha_k E_k - \alpha_k.\end{aligned}$$

(We have used the homogeneity restriction, the Engel aggregation and the fact that $\Sigma_i \alpha_i = 1$.) Hence,

$$\sum_i e_{ik}\alpha_i = -\alpha_k.$$

The Cournot aggregation suggests an interesting question. Can we be sure that the triad contains all the useful restrictions which flow from the utility-maximizing model in which the utility function is strictly quasiconcave and differentiable, but otherwise unrestricted? Even though we have found that the Cournot aggregation is implied by the triad, perhaps there are other restrictions which are independent of the triad? The answer is that apart from some restrictions on the signs of various price elasticities (see E2.10), the triad *does* summarize the complete set of restrictions on the demand elasticities. This is a principal conclusion from the literature on the so-called *integrability* problem. But since integrability is rather a difficult topic, we will delay consideration of it until Chapter 3.

Exercise 2.9. The Hicks–Slutsky partition

Show that for all i and j,

$$\frac{\partial x_i}{\partial p_j} = \left(\frac{\partial x_i}{\partial p_j}\right)_{\mathrm{d}U=0} - x_j\frac{\partial x_i}{\partial y} \tag{E2.9.1}$$

or, equivalently,

$$e_{ij} = e_{ij}^{\mathrm{c}} - E_i\alpha_j, \tag{E2.9.2}$$

where $(\partial x_i/\partial p_j)_{\mathrm{d}U=0}$ is the *compensated* derivative of the demand for good i with respect to changes in price j, i.e. $\partial x_i/\partial p_{j\,\mathrm{d}U=0}$ is the effect on the demand for good i when there is a change in both p_j and a change in y which is sufficient to allow the consumer to maintain his initial level of utility. e_{ij}^{c} is the compensated elasticity, i.e.

$$e_{ij}^{\mathrm{c}} = \frac{p_j}{x_i}\left(\frac{\partial x_i}{\partial p_j}\right)_{\mathrm{d}U=0}.$$

Hint: the first step is to find the change in income, $\mathrm{d}y$, which is necessary to compensate for a change, $\mathrm{d}p_j$, in price j. This can be done by noting that

$$\mathrm{d}U = \sum_k \frac{\partial U}{\partial x_k}\,\mathrm{d}x_k = \sum_k \lambda p_k\,\mathrm{d}x_k = 0 \tag{E2.9.3}$$

and

$$\mathrm{d}y = \sum_k p_k\,\mathrm{d}x_k + (\mathrm{d}p_j)x_j, \tag{E2.9.4}$$

where dU is the change in utility and the dx_k, dy are the changes in consumption levels and income arising from the compensated change in p_j. λ is the marginal utility of income

Answer. From (E2.9.3) and (E.2.9.4) we see that the compensation, dy, necessary to allow the consumer to retain his initial level of utility is given by

$$dy = (dp_j)x_j.$$

Therefore the change in x_i arising from both a change in p_j and a utility compensating change in income is

$$\left(dx_i\right)_{dU=0} = \frac{\partial x_i}{\partial p_j}\, dp_j + \frac{\partial x_i}{\partial y}\, (dp_j)x_j,$$

i.e.

$$\left(\frac{\partial x_i}{\partial p_j}\right)_{dU=0} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial y}\, x_j. \tag{E2.9.5}$$

This equation can quickly be rearranged to give (E2.9.1) and (E2.9.2).

The Hicks–Slutsky partition divides the total effect on x_i of a change in p_j into two parts. Referring to (E2.9.1), the first term on the right is the substitution effect. In terms of the usual diagram,[17] the substitution effect arises from the movement around the initial indifference curve. The second term is the *income* effect. It captures the idea that an uncompensated change in p_j will affect the consumer's purchases of good i by affecting the real value of his total budget.

Exercise 2.10. The negativity of the own-price substitution effect

Demonstrate that

$$\left(\frac{\partial x_i}{\partial p_i}\right)_{dU=0} \leqslant 0, \quad \text{for all } i.$$

This can be done by appealing to the second-order conditions, see for example Lancaster (1968, pp. 56–58). However, a simpler and more direct argument is available using ideas from the theory of revealed preference, see for example Lancaster (1968, pp. 125–127) or Baumol (1972, pp. 231–232).

Answer. We assume that the consumer chooses his consumption vector x to maximize a strictly quasiconcave, differentiable utility function, $U(x)$, subject to

[17] See, for example, Intriligator (1971, p. 161).

his budget constraint $p'x = y$.[18] We consider two situations in both of which the consumer achieves the same utility level. The two situations can be thought of as the initial and final situations after a compensated change in prices. In the first situation the price vector is $\bar{p}$ and income is $\bar{y}$. In the second, the price vector is $\bar{\bar{p}}$, not equal to $\bar{p}$, and income is $\bar{\bar{y}}$.[19] Where $\bar{x}$ and $\bar{\bar{x}}$ are the optimum consumption vectors for each situation, we have

$$U(\bar{x}) = U(\bar{\bar{x}}),$$

with $\bar{x} \neq \bar{\bar{x}}$.

Fig. E2.10.1 illustrates the two situations for the two-good case. It will be noticed that the indifference curve is 'smooth' with no linear segments or 'corners' and with the right convexity. The *strict* quasiconcavity assumption rules out the possibility of linear segments and the differentiability assumption excludes indifference curves of the type shown in fig. E2.10.2. (After you have understood the argument given here, you might work out what modifications are necessary if the utility function is not strictly quasiconcave or has nondifferentiable points.)

We note that

$$\bar{p}'\bar{\bar{x}} > \bar{y} = \bar{p}'\bar{x} \tag{i}$$

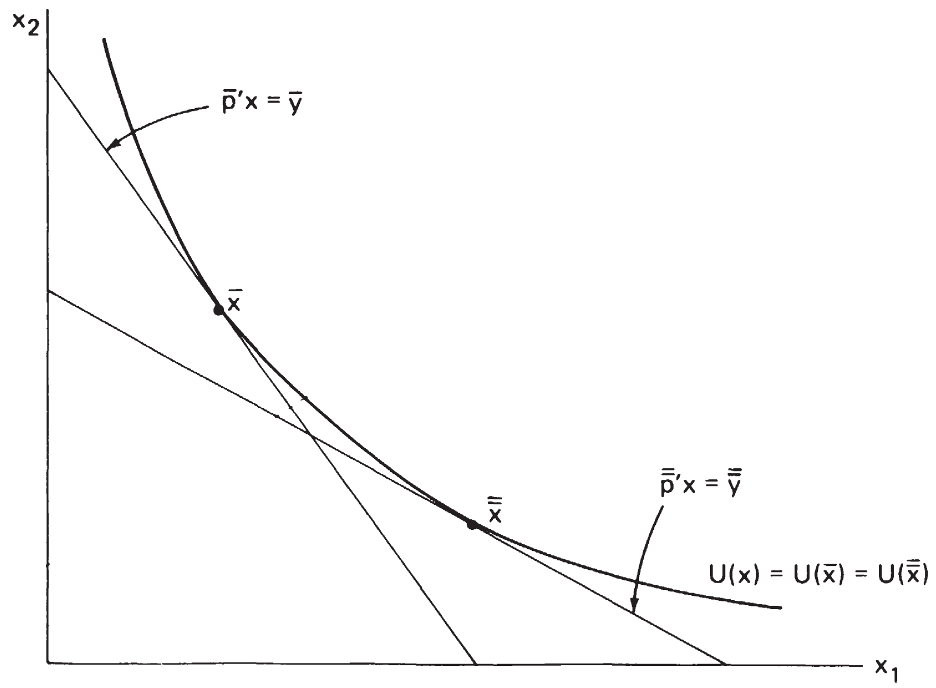

Figure E2.10.1

[18] We assume that the consumer spends his entire budget.

[19] We assume that relative prices have changed, i.e. $\bar{p} \neq \beta\bar{\bar{p}}$, where β is a scalar.

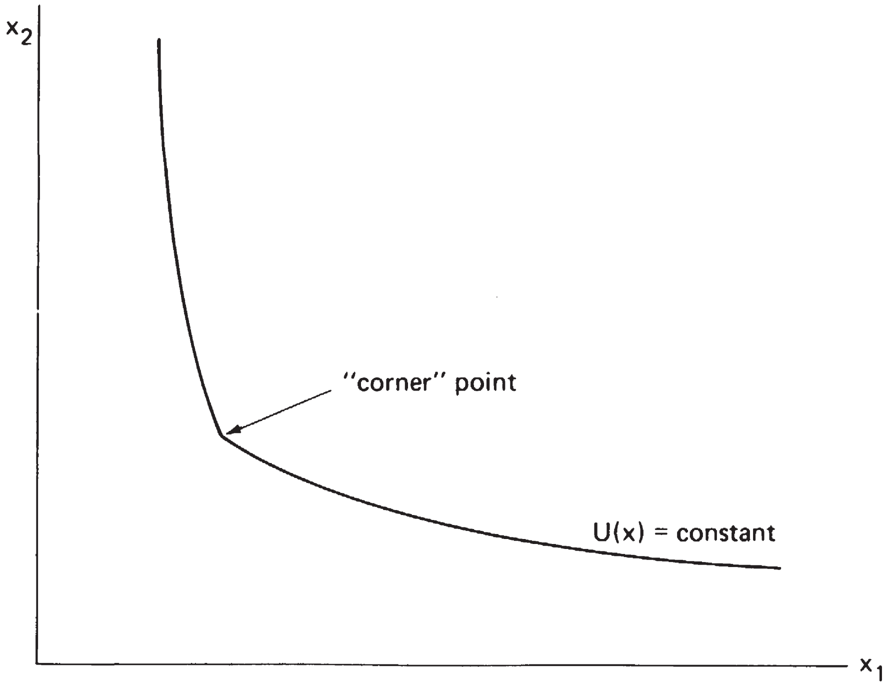

Figure E2.10.2

and

$$\bar{\bar{p}}'\,\bar{x} > \bar{\bar{y}} = \bar{\bar{p}}'\,\bar{\bar{x}}. \tag{ii}$$

The first of these inequalities means that when the consumer was faced with the initial prices, $\bar{p}$, and income, $\bar{y}$, he could not afford the consumption vector $\bar{\bar{x}}$. If $\bar{p}'\,\bar{\bar{x}}$ were equal to or less than $\bar{y}$, then $\bar{x}$ and $\bar{\bar{x}}$ would be alternative optima for the initial consumer problem. However, the assumption of *strict* quasiconcavity implies that the consumer problem has a unique solution. The justification for inequality (ii) is similar to that for (i).

From (i) and (ii) we find that

$$\bar{p}'\,(\bar{\bar{x}} - \bar{x}) > 0$$

and

$$\bar{\bar{p}}'\,(\bar{x} - \bar{\bar{x}}) > 0.$$

Hence

$$\bar{p}'\,(\bar{\bar{x}} - \bar{x}) + \bar{\bar{p}}'\,(\bar{x} - \bar{\bar{x}}) > 0,$$

i.e.

$$(\bar{\bar{p}}' - \bar{p}')\,(\bar{x} - \bar{\bar{x}}) > 0$$

and

$$(\bar{\bar{p}}' - \bar{p}')(\bar{\bar{x}} - \bar{x}) < 0.$$

This final inequality can be written as

$$(\mathrm{d}p)'(\mathrm{d}x) < 0, \tag{iii}$$

where dp and dx are the vectors of price and consumption changes between the final and initial situations. In the special case where only price i changes, then

$$(\mathrm{d}p_i)(\mathrm{d}x_i) < 0.$$

Hence a compensated increase in p_i produces a reduction in x_i; $(\mathrm{d}p_i)$ and $(\mathrm{d}x_i)$ are of opposite signs. We can conclude that

$$\left(\frac{\partial x_i}{\partial p_i}\right)_{\mathrm{d}U=0} \leqslant 0. \tag{E2.10.1}$$

It is usual to conclude that

$$\left(\frac{\partial x_i}{\partial p_i}\right)_{\mathrm{d}U=0} < 0.$$

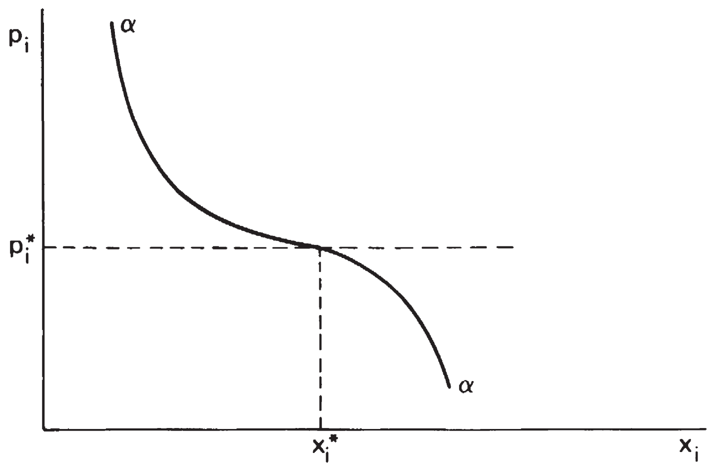

Figure E2.10.3. $\alpha\alpha$ is the compensated demand curve for good i, i.e. it shows the demand for good i as we vary p_i holding all other prices and the utility level constant. The curve is consistent with having

$$(\mathrm{d}p_i)(\mathrm{d}x_i) < 0.$$

But it is also true that

$$\left(\frac{\partial x_i}{\partial p_i}\right)_{\mathrm{d}U=0} = 0$$

at the point where $x_i = x_i^*$.

Our argument, however, has not explicitly ruled out the case illustrated in fig. E2.10.3. Hence, we will be content with (E2.10.1).

For future reference (see E3.14) it will be useful to note that in addition to (E2.10.1), (iii) implies that the matrix of compensated own and cross price derivatives is negative semidefinite. (A symmetric matrix A is said to be negative semidefinite if $x'Ax \leqslant 0$ for all choices of the vector x.) To obtain this result we write

$$\mathrm{d}x = G\mathrm{d}p, \tag{iv}$$

where $\mathrm{d}x$ is the vector of changes in consumption arising from a compensated change in prices $\mathrm{d}p$, and G is the matrix of compensated own and cross price derivatives,[20] i.e.

$$G_{ij} = \left(\frac{\partial x_i}{\partial p_j}\right)_{\mathrm{d}U=0}, \quad \text{for all} \quad i, j.$$

If $\mathrm{d}p = \beta p$, where β is a scalar, then

$$\mathrm{d}x = 0. \tag{v}$$

(A compensated proportionate change in all prices leaves consumption unchanged.) If $\mathrm{d}p \neq \beta p$, then according to (iii) we have

$$(\mathrm{d}p)'\mathrm{d}x < 0. \tag{vi}$$

On combining (iv)–(vi) we see that

$$(\mathrm{d}p)'G\mathrm{d}p \leqslant 0 \tag{E2.10.2}$$

for all choices of $\mathrm{d}p$.

Exercise 2.11. The inferiority of Giffen goods

Are all Giffen goods inferior?

Answer. For Giffen goods we have

$$0 < \frac{\partial x_i}{\partial p_i} = \left(\frac{\partial x_i}{\partial p_i}\right)_{\mathrm{d}U=0} - x_i \frac{\partial x_i}{\partial y}.$$

[20] G is the matrix of derivatives whose typical element is defined by (E2.9.5). Notice that G is symmetric, i.e.

$$\left(\frac{\partial x_i}{\partial p_j}\right)_{\mathrm{d}U=0} = \left(\frac{\partial x_j}{\partial p_i}\right)_{\mathrm{d}U=0}$$

This follows from (E2.9.5) and the symmetry restriction – see in particular eq. (viii) in (E2.6).

Given the nonpositivity of the own-price substitution effect, the above inequality implies that

$$\frac{\partial x_i}{\partial y} < 0.$$

Hence, Giffen goods are inferior.

Exercise 2.12. The sign of cross-elasticities of demand

Although two goods i and j may be substitutes, the cross-elasticity of demand for i with respect to changes in the price of j can be negative. Explain with the aid of a suitable diagram.

Answer. Two goods i and j are said to be substitutes if and only if

$$\left(\frac{\partial x_i}{\partial p_j}\right)_{\mathrm{d}U=0} \geqslant 0. \qquad \text{(i)}$$

Inequality (i) may also be written as

$$e_{ij}^{\mathrm{c}} \geqslant 0. \qquad \text{(ii)}$$

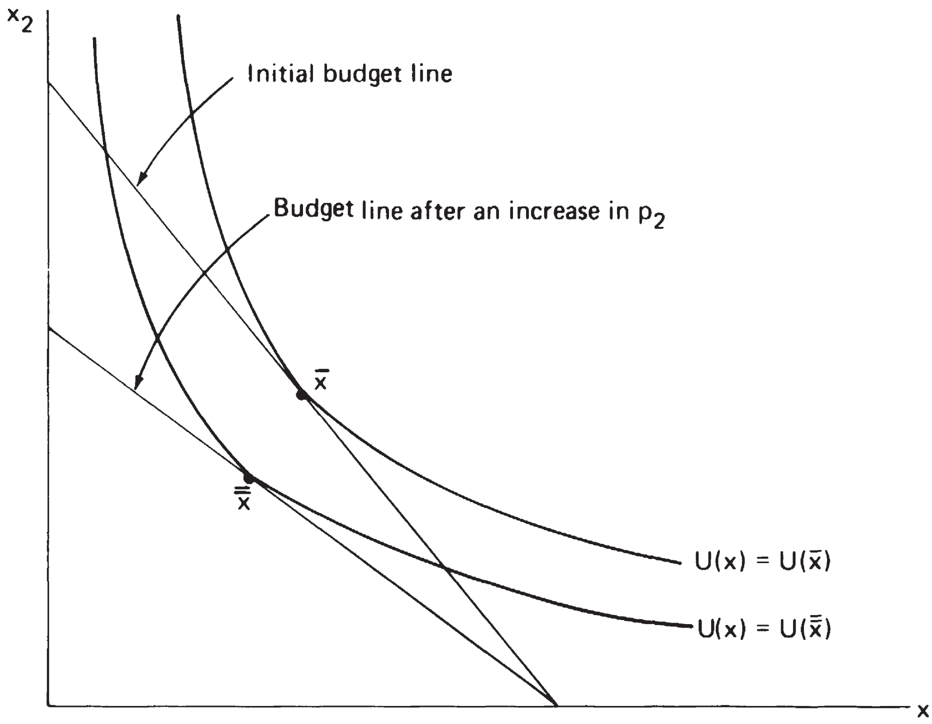

Figure E2.12.1. An increase in p_2 moves the consumer equilibrium from $\bar{x}$ to $\bar{\bar{x}}$. In particular an increase in p_2 *reduces* x_1, i.e. $e_{12} < 0$.

The Hicks–Slutsky partition implies that

$$e_{ij} = e^{c}_{ij} - \alpha_j E_i$$

(see (E2.9.2)).

If $\alpha_j E_i > e^{c}_{ij}$, then despite (ii), e_{ij} will be negative. Fig. E2.12.1 illustrates a situation, for the two-good model, in which goods 1 and 2 are substitutes (in view of E2.13, they could not be anything else), yet the cross-elasticity of demand e_{12} is negative.

Exercise 2.13. Substitutes, the two-good case

Show that in the two-good case, all goods must be substitutes.

Answer. For the two-good case we consider a compensated increase in p_1 with p_2 constant. Then the changes in x_1 and x_2 are

$$\mathrm{d}x_1 = \left(\frac{\partial x_1}{\partial p_1}\right)_{\mathrm{d}U=0} \mathrm{d}p_1$$

and

$$\mathrm{d}x_2 = \left(\frac{\partial x_2}{\partial p_1}\right)_{\mathrm{d}U=0} \mathrm{d}p_1 .$$

The change in utility is given by

$$\mathrm{d}U = \frac{\partial U}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial U}{\partial x_2}\,\mathrm{d}x_2 = 0.$$

Hence

$$\left(\frac{\partial U}{\partial x_1}\right)\left(\frac{\partial x_1}{\partial p_1}\right)_{\mathrm{d}U=0} \mathrm{d}p_1 + \left(\frac{\partial U}{\partial x_2}\right)\left(\frac{\partial x_2}{\partial p_1}\right)_{\mathrm{d}U=0} \mathrm{d}p_1 = 0. \tag{i}$$

We assume that $\partial U/\partial x_1$ and $\partial U/\partial x_2$ are positive. In fact, they are equal to the marginal utility of income multiplied by the respective prices. We also know that the own-price substitution effect is nonpositive. We may conclude from (i), that

$$\left(\frac{\partial x_2}{\partial p_1}\right)_{\mathrm{d}U=0} \geqslant 0.$$

Hence goods 1 and 2 are substitutes.

Exercise 2.14. Consumer behavior under rationing

Assume that a consumer has the utility function

$$U(x_1, x_2) = x_1 x_2,$$

where x_1 and x_2 represent the amounts of two goods, 1 and 2, consumed in a given time period. Find his utility-maximizing consumption levels subject to the budget constraint

$$5x_1 + 4x_2 \leqslant 50 \quad (p_1 = 5, p_2 = 4 \text{ and } y = 50).$$

Now suppose that a rationing system is imposed on the consumer. The ration point 'prices' of x_1 and x_2 are 3 and 6, respectively, and the consumer is issued a total of 40 ration points. Find his optimum consumption levels. Are both the ration and budget constraints binding? What is the effect on the consumer's utility of having respectively one more ration point and one more dollar.

Demonstrate that the existence of a market for ration points will improve the consumer's welfare. In particular, calculate the consumer's utility, optimum consumption levels, and the number of ration points which he buys or sells when the price of a ration point is $p_3 = 0.5$.

Answer. In the initial situation, i.e. before the imposition of rationing, the consumer will choose non-negative values for x_1 and x_2 to maximize

$$x_1 x_2$$

subject to

$$5x_1 + 4x_2 \leqslant 50.$$

We may assume that the constraint is binding and that the optimal values for x_1 and x_2 are strictly positive. Thus, the first-order conditions for a solution are

$$\left.\begin{aligned} x_2 - 5\lambda &= 0, \\ x_1 - 4\lambda &= 0 \\ 5x_1 + 4x_2 &= 50. \end{aligned}\right\} \qquad \text{(i)}$$

From (i), we find that

$$x_1 = 5, \quad x_2 = 6\tfrac{1}{4} \quad \text{and} \quad \lambda = 1\tfrac{1}{4}.$$

With the imposition of rationing, the consumer's problem becomes that of choosing non-negative values for x_1 and x_2 to maximize

$$x_1 x_2 \qquad \text{(ii)}$$

subject to

$$5x_1 + 4x_2 \leqslant 50$$

and

$$3x_1 + 6x_2 \leqslant 40.$$

It is obvious that the optimal values of x_1 and x_2 will be strictly positive. On the other hand, it is not obvious which of the constraints will be binding. Hence we cannot take any short cuts.

On applying the Lagrangian method to generate the first-order conditions, we find that at a solution for problem (ii) there will exist non-negative values for λ_1 and λ_2 such that

$$\left.\begin{aligned} x_2 - 5\lambda_1 - 3\lambda_2 &= 0, \\ x_1 - 4\lambda_1 - 6\lambda_2 &= 0, \\ 5x_1 + 4x_2 - 50 &\leqslant 0, \\ 3x_1 + 6x_2 - 40 &\leqslant 0, \\ \lambda_1 (5x_1 + 4x_2 - 50) &= 0, \\ \lambda_2 (3x_1 + 6x_2 - 40) &= 0. \end{aligned}\right\} \quad \text{(iii)}$$

Certainly, not both λ_1 and λ_2 can be zero. This would imply that x_1 and x_2 are both zero, giving a value of zero for the objective function. There is no doubt that the consumer can do better than that.

Can we have $\lambda_2 = 0$ with $\lambda_1 > 0$? If $\lambda_1 > 0$, then the first constraint holds as an equality, i.e.

$$5x_1 + 4x_2 - 50 = 0.$$

Also, with $\lambda_2 = 0$ the first two equations in (iii) imply that

$$5x_1 - 4x_2 = 0.$$

From here we obtain

$$x_1 = 5 \quad \text{and} \quad x_2 = 6\tfrac{1}{4}.$$

However, these values for x_1 and x_2 violate the second constraint.

Can we have $\lambda_1 = 0$ with $\lambda_2 > 0$? Under these conditions, we find that

$$3x_1 + 6x_2 = 40$$

and

$$x_1 - 2x_2 = 0.$$

This gives

$$x_1 = 6\tfrac{2}{3}; \quad x_2 = 3\tfrac{1}{3},$$

and these values of x_1 and x_2, together with

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 1\tfrac{1}{9}$$

satisfy (iii).

The final case to be checked is $\lambda_1 > 0$, $\lambda_2 > 0$. This would imply that both constraints were binding, giving

$$x_1 = \tfrac{70}{9} \quad \text{and} \quad x_2 = \tfrac{25}{9}.$$

Then from the first two equations in (iii) we would have

$$\tfrac{25}{9} = 5\lambda_1 + 3\lambda_2$$

and

$$\tfrac{70}{9} = 4\lambda_1 + 6\lambda_2,$$

i.e.

$$\lambda_1 = -\tfrac{10}{27} \quad \text{and} \quad \lambda_2 = \tfrac{125}{81}.$$

Negative values for the λ's are not allowed. We may conclude that the problem solution occurs in the second case examined, i.e. with $\lambda_1 = 0$ and $\lambda_2 > 0$, giving $x_1 = 6\tfrac{2}{3}$ and $x_2 = 3\tfrac{1}{3}$.

The marginal utilities of dollars and ration points are given by the Lagrangian multipliers. An additional dollar generates no utility. (Notice that the consumer is not using all his dollar budget.) On the other hand, an additional ration point will yield $1\tfrac{1}{9}$ units of utility.

It is now clear that the consumer must benefit from the existence of a market for ration points. At the margin, dollars are of no value to him. If he can trade dollars for ration points, he will be able to achieve a higher level of utility. In particular, if ration points can be bought and sold for $\$\,\tfrac{1}{2}$, then he is free to choose non-negative values for x_1 and x_2 to maximize

$$x_1 x_2 \tag{iv}$$

subject to

$$(5 + 1\tfrac{1}{2})x_1 + (4 + 3)x_2 \leqslant 50 + 20.$$

The price of good 1 is \$ 5 plus 3 ration points, or effectively \$ $6\tfrac{1}{2}$. The price of good 2 is \$ 4 plus 6 ration points, or effectively \$ 7. The consumer's budget is \$ 50 plus 40 ration points, or effectively \$ 70. Alternatively we could work in ration points. Then the budget constraint would be

$$(10 + 3)x_1 + (8 + 6)x_2 \leqslant 100 + 40.$$

The price of good 1 would be 13 ration points, etc.

Proceeding with problem (iv) we find that the first-order conditions are

$$x_2 - 6\tfrac{1}{2}\lambda = 0,$$

$$x_1 - 7\lambda = 0$$

and

$$6\tfrac{1}{2}x_1 + 7x_2 = 70,$$

and thus

$$x_1 = \tfrac{70}{13} \quad \text{and} \quad x_2 = 5.$$

With the existence of the market for ration points, the consumer has increased his utility from

$$U_1 = (6\tfrac{2}{3})(3\tfrac{1}{3}) = 22\tfrac{2}{9},$$

where U_1 is his initial utility under rationing, to

$$U_2 = (\tfrac{70}{13})(5) = 26\tfrac{12}{13}.$$

He has done this by trading dollars for ration points. He now uses

$$(\tfrac{70}{13})(3) + (5)(6) = 46\tfrac{2}{13}$$

ration points. Of his \$ 50 budget, he has used

$$(6\tfrac{2}{13})(\tfrac{1}{2}) = \$\ 3\tfrac{1}{13}$$

to buy additional ration points and

$$(5)(\tfrac{70}{13}) + (4)(5) = \$\ 46\tfrac{12}{13}$$

to meet the dollar prices of his purchases of goods 1 and 2.

Exercise 2.15. The allocation of time

A self-sufficient farmer lives on produce he grows himself under conditions of diminishing marginal productivity of labor. The length of his working time can be explained in terms of a utility-maximizing choice between agricultural produce and leisure. There is a minimum real hourly wage rate which could just induce him to quit farming and become a hired worker. If this minimum wage were offered to him, and he became a hired worker, would the length of his working time (1) remain the same, (2) become shorter, or (3) become longer than when he was a self-sufficient farmer?

What institutional assumptions are implicit in your response? Note that for the farmer, who owns all the necessary means of production, the labor–leisure choice is a real possibility. Under what conditions will the wage laborer have control over his hours of work?

Answer. The solution to this problem can be seen immediately in fig. E2.15.1. *uu* is an indifference curve for the farmer's produce–leisure choice; *LP* is his consumption possibilities curve while he remains self-employed. (What is the relationship of *LP* to the total product of labor curve?) The curvature of *LP* exhibits diminishing marginal product. The farmer's utility-maximizing combination of leisure and product is at *A*. If the farmer becomes a wage-earner, his consumption possibilities curve changes to a straight line; i.e. we are assuming that the farmer is paid a fixed amount of 'product' per hour. The slope of *LB* represents the minimum hourly wage rate which could induce him to quit farming; the corresponding consumption possibilities line *LB* just allows him to reach *uu*. It is clear that as a wage-earner the farmer will work longer than when he was self-employed. His utility-maximizing bundle is at *B*, giving him more product but less leisure than at *A*.

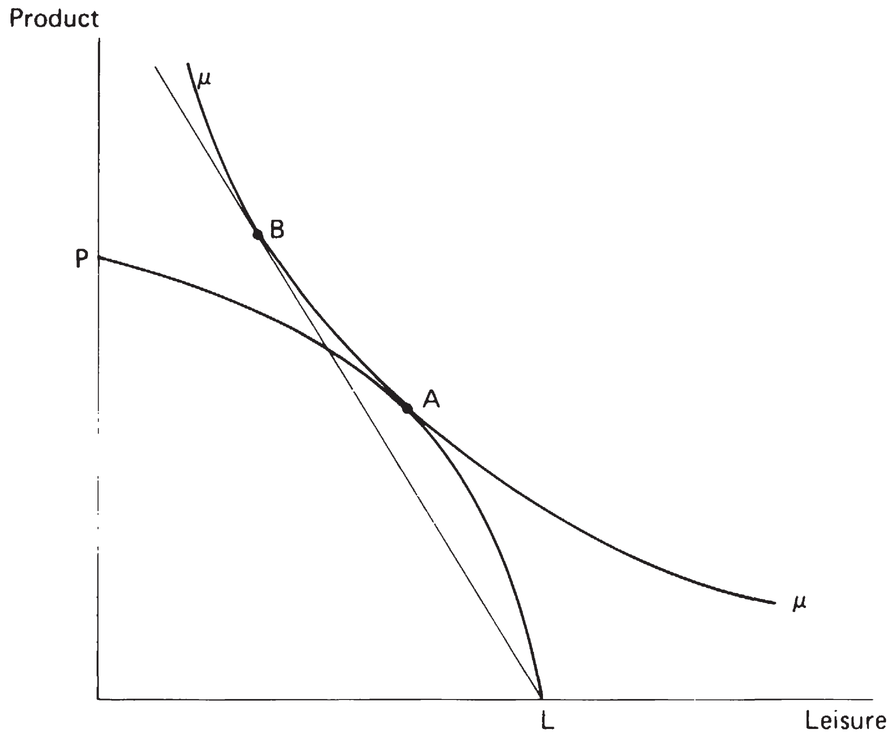

Figure E2.15.1

Be sure you understand why the answer depends crucially on the fact that for the self-employed farmer the marginal rate of transformation of leisure into product is diminishing, but when he hires his labor out on the market, he faces a constant marginal rate of transformation of leisure into product.

Exercise 2.16. Pure exchange and Pareto optimality

Consider an economy of two men who consume just two goods, x and y. Man 1 has an intitial endowment of 40 units of x and 160 of y while man 2 has 240 and 120 units of x and y, respectively. Assume that the men have the following utility functions:

$$U_1 = x_1 y_1 = \text{utility of man 1}$$

$$U_2 = x_2 y_2^2 = \text{utility of man 2,}$$

where x_i and y_i are the amounts of goods x and y consumed by man i.

(a) Using the standard geometrical apparatus (the 'Edgeworth–Bowley box'), show that in general there exists some exchange (trade) between the two which will result in their both obtaining a higher level of utility.

(b) We define a situation to be *efficient* or *Pareto optimal* if there is no reallocation of commodities between the two men which gives one of them higher utility without lowering the utility of the other. Show that a necessary condition for efficiency is

$$\frac{\partial U_1}{\partial x_1} \bigg/ \frac{\partial U_1}{\partial y_1} = \frac{\partial U_2}{\partial x_2} \bigg/ \frac{\partial U_2}{\partial y_2} .$$

Is the assumption of efficiency sufficient to determine what trading will be done between man 1 and man 2?

(c) Under the assumption of competition, each man faces a common set of prices p_x and p_y. Each maximizes his utility subject to his budget constraint and prices adjust so that supply equals demand for all goods. Determine the competitive values for x_1, y_1, x_2, y_2 and p_y. We may assume that $p_x = 1$. (Why?) If the algebra becomes a little tedious, you may prefer to limit yourself to writing down the relevant equations and checking that $x_1 = 140, x_2 = 140, y_1 = 93\frac{1}{3}, y_2 = 186\frac{2}{3}$ and $p_y = 1\frac{1}{2}$ is the solution. Is the competitive allocation of goods Pareto optimal?

Answer.

(a) Consider fig. E2.16.1. A represents the initial endowments. Both men gain welfare by any trade which moves them into the shaded area.

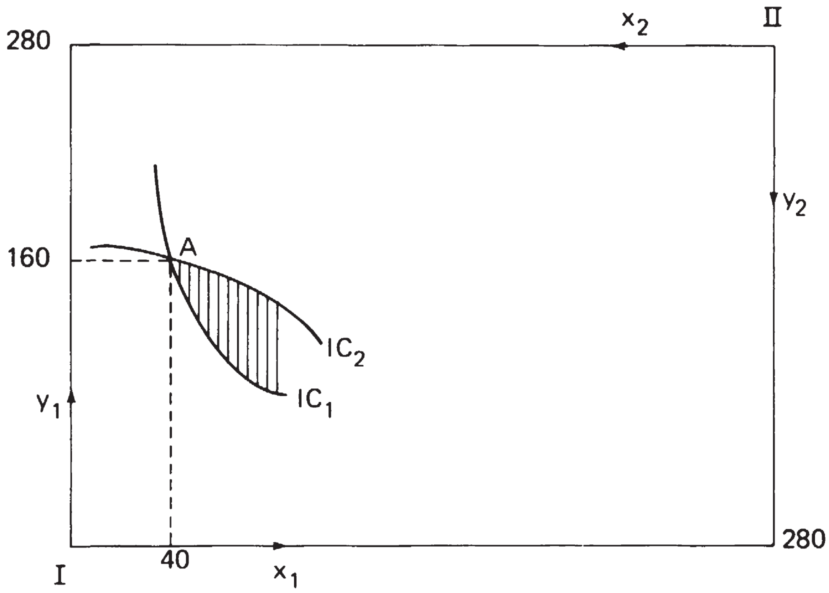

Figure E2.16.1

(b) Goods x and y can be allocated efficiently only if x_1, y_1, x_2 and y_2 maximize

$$U_1(x_1, y_1) \tag{i}$$

subject to

$$\bar{U}_2 = U_2(x_2, y_2),$$

$$x_1 + x_2 = \bar{x}_1 + \bar{x}_2$$

and

$$y_1 + y_2 = \bar{y}_1 + \bar{y}_2,$$

where $\bar{U}_2$ is a predetermined level of utility for consumer 2 and the $\bar{x}_i, \bar{y}_i$ are the initial endowments. In other words, given the utility level of man 2, efficiency implies that the available goods must be allocated so that the utility level for man 1 is a maximum. Alternatively, we could work with a problem in which man 2's utility is maximized subject to achieving a given level of utility for man 1 and subject to the availability of goods.

Necessary conditions for the solution of problem (i) can be generated by differentiating the Lagrangian,

$$L = U_1(x_1, y_1) - \lambda(\bar{U}_2 - U_2(x_2, y_2))$$
$$- \Pi_x(x_1 + x_2 - \bar{x}_1 - \bar{x}_2) - \Pi_y(y_1 + y_2 - \bar{y}_1 - \bar{y}_2),$$

where λ, Π_x and Π_y are the Lagrangian multipliers. Hence, for an efficient allocation, we will have

$$\frac{\partial U_1}{\partial x_1} - \Pi_x = 0,$$

$$\frac{\partial U_1}{\partial y_1} - \Pi_y = 0,$$

$$\lambda \frac{\partial U_2}{\partial x_2} - \Pi_x = 0$$

and

$$\lambda \frac{\partial U_2}{\partial y_2} - \Pi_y = 0,$$

as well as the initial constraints. By carrying out the obvious divisions we find that

$$\frac{\Pi_x}{\Pi_y} = \frac{\partial U_1}{\partial x_1} \Big/ \frac{\partial U_1}{\partial y_1} = \frac{\partial U_2}{\partial x_2} \Big/ \frac{\partial U_2}{\partial y_2} . \tag{E2.16.1}$$

Efficiency alone is not sufficient to determine the trading pattern between man 1 and man 2. Efficiency merely locates us on the *contract curve*, see fig. E2.16.2. Different points on the contract curve may be generated by changing

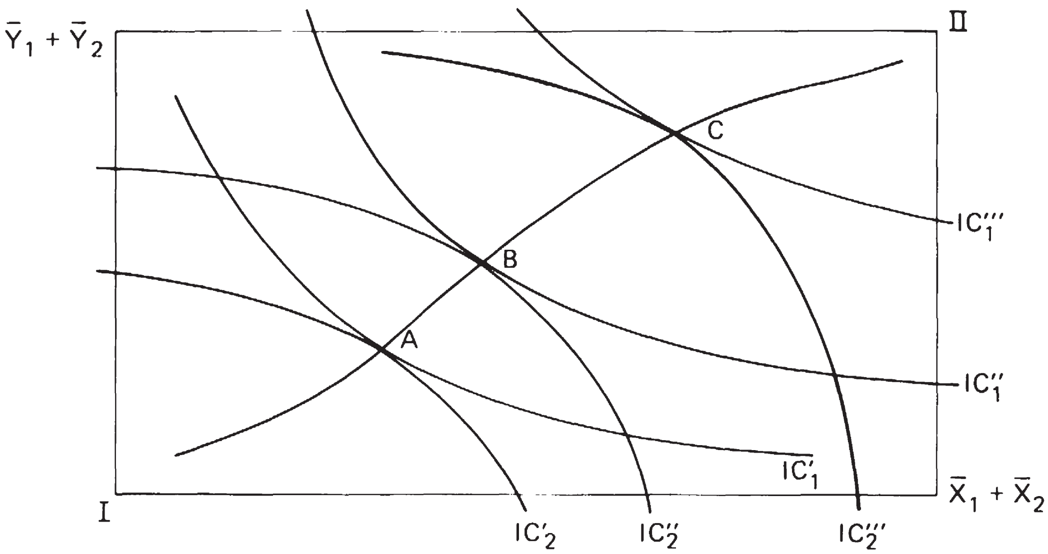

Figure E2.16.2. IC_1', IC_1'' and IC_1''' are three indifference curves for man 1 and IC_2', IC_2'' and IC_2''' are three indifference curves for man 2. The contract curve or locus of efficient points, i.e. those satisfying (E2.16.1), is shown as A, B, C.

the value of $\bar{U}_2$ in problem (i). As we increase $\bar{U}_2$ the solution of problem (i) moves along the contract curve towards I. To attain a unique solution for the trading problem, we need some additional information or restrictions. These might be provided by having some rule which allows us to evaluate alternative distributions of utility (a social welfare function) and to choose the one we like best. But in the absence of a social welfare function (or some other device), the indeterminacy remains.

(c) Man 1 chooses x_1 and y_1 to maximize

$$x_1 y_1$$

subject to

$$p_x x_1 + p_y y_1 = p_x 40 + p_y 160.$$

The first-order conditions are

$$y_1 = \lambda p_x,$$

$$x_1 = \lambda p_y$$

and

$$p_x x_1 + p_y y_1 = p_x 40 + p_y 160.$$

On eliminating λ we obtain

$$x_1 = \left(\frac{p_y}{p_x}\right) y_1 \tag{ii}$$

and

$$x_1 + \left(\frac{p_y}{p_x}\right) y_1 = 40 + 160\left(\frac{p_y}{p_x}\right).^{21} \tag{iii}$$

Similarly we find that for man 2,

$$2x_2 = \left(\frac{p_y}{p_x}\right) y_2 \tag{iv}$$

and

$$x_2 + \left(\frac{p_y}{p_x}\right) y_2 = 240 + 120\left(\frac{p_y}{p_x}\right). \tag{v}$$

[21] Eq. (iii) is simply a rearrangement of the budget constraint. This form will be convenient for our explanation of why the competitive equilibrium determines relative prices, but not absolute prices.

Finally, we note that total consumption must equal the total supplies, i.e.

$$x_1 + x_2 = 280\ (= \bar{x}_1 + \bar{x}_2) \tag{vi}$$

and

$$y_1 + y_2 = 280\ (= \bar{y}_1 + \bar{y}_2). \tag{vii}$$

To summarize, we have found that x_1, x_2, y_1, y_2 and p_y/p_x are consistent with the competitive equilibrium only if they satisfy eqs. (ii)–(vii). Notice that relative prices, (p_y/p_x), not absolute prices, appear in the equations. Hence we can assume that $p_x = 1$ (or any other positive number) and determine p_y. Also, it is worth pointing out that any one of the equations (iii), (v), (vi) and (vii) can be derived from the remaining three. For example, we can derive (vii) from (iii), (v) and (vi). Rewriting (iii), (v) and (vi) with the notation $k = p_y/p_x$, we have

$$x_1 + ky_1 = 40 + 160k,$$

$$x_2 + ky_2 = 240 + 120k$$

and

$$x_1 + x_2 = 280.$$

Adding the first two of these equations and using the third, we find that

$$280 + ky_1 + ky_2 = 280 + 160k + 120k,$$

i.e.

$$y_1 + y_2 = 280,$$

which is precisely eq. (vii). Hence in determining the competitive values for the five variables x_1, x_2, y_1, y_2 and k, we need use only the five equations (ii)–(vi). The last of the supply and demand equations will be satisfied automatically if we find a solution for the first five equations.[22]

To solve the five-equation system (ii)–(vi) we proceed as follows: first we substitute from (ii) and (iv) into (iii), (v) and (vi), to eliminate x_1 and x_2. This gives.

$$2ky_1 = 40 + 160k,$$

$$1\tfrac{1}{2}\,ky_2 = 240 + 120k$$

and

$$ky_1 + \tfrac{1}{2}ky_2 = 280.$$

[22] Our elimination of one of the demand-equals-supply equations is an example of the application of Walras's law.

Next we eliminate y_1, reducing our system to

$$1\tfrac{1}{2}ky_2 - 120k = 240$$

and

$$80k + \tfrac{1}{2}ky_2 = 260.$$

Finally, we eliminate k and we find that

$$\frac{1\frac{1}{2}y_2 - 120}{80 + \frac{1}{2}y_2} = \frac{240}{260}.$$

Hence

$$y_2 = 186\tfrac{2}{3}$$

$$y_1 = \ \ 93\tfrac{1}{3}$$

$$x_1 = 140$$

$$x_2 = 140$$

and

$$k \ = 1\tfrac{1}{2} = (p_y/p_x).$$

The competitive mechanism resolves the indeterminacy discussed in part (b). Under competitive conditions, each man, i, organizes his purchases so that

$$\frac{\partial U_i}{\partial x_i} \Big/ \frac{\partial U_i}{\partial y_i} = \frac{p_x}{p_y}.$$

Thus (E2.16.1) is satisfied and the competitive equilibrium is Pareto optimal. Compared with part (b), we have the additional information that consumers must satisfy their budgets. This is sufficient to determine at which point on the contract curve the equilibrium will occur.[23]

Exercise 2.17. Additive utility functions

A popular assumption in empirical work is that the utility function is additive, i.e. it can be written in the form

$$U(x_1,...,x_n) = U^1(x_1) + U^2(x_2) + ... + U^n(x_n),$$

[23] In a more general example, it is possible that there are several competitive equilibria. Hence, even the adoption of competitive assumptions may not completely eliminate the indeterminacy of the pure exchange model.

where each of U^1, U^2,..., U^n is a strictly concave function of a single argument. The log-linear utility function, (E2.3.3), is an example. Under the additivity assumption, the symmetry condition (E2.6.2) can be replaced by the stronger restriction

$$e_{ij} = -E_i \alpha_j \left(1 + \frac{E_j}{\omega}\right), \quad \text{for all } i \neq j, \tag{E2.17.1}$$

where ω is a scalar, independent of i and j. ω is often referred to as the 'Frisch parameter'.

(a) Is an additive utility function likely to be an adequate description of consumer preferences in a very detailed model where the commodity classifications include, for example, fruit, vegetables, meat, fish, cotton shirts, synthetic shirts, etc.? Would the additivity assumption be more easily sustainable in an aggregative model based on commodity classifications such as food, clothing, etc.?

(b) Check that (E2.17.1) is in fact a more severe restriction than (E2.6.2), i.e. check that (E2.17.1) implies (E2.6.2), but not vice versa.

(c) The derivation of (E2.17.1) is fairly time consuming. However, it illustrates a general method of translating restrictions on the utility function (in this case the zeros in the Hessian) into empirically useful restrictions on the demand elasticities. As a first step, prove that

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix}^{-1} = \frac{1}{p' H^{-1} p} \left[\begin{array}{c|c} (p'H^{-1}p)H^{-1} - H^{-1}pp'H^{-1} & H^{-1}p \\ \hline p'H^{-1} & -1 \end{array}\right]. \tag{E2.17.2}$$

The relevance of this step will be apparent from (E2.6.1). Once you have established (E2.17.2), you will be able to show that

$$\frac{\partial \lambda}{\partial y} = \frac{1}{p'H^{-1}p}, \tag{E2.17.3}$$

$$\frac{\partial x_i}{\partial y} = (H^{-1}p)_i \frac{\partial \lambda}{\partial y} \tag{E2.17.4}$$

and

$$\frac{\partial x_i}{\partial p_j} = (H^{-1})_{ij}\lambda - \frac{\partial \lambda}{\partial y}(H^{-1}p)_i (H^{-1}p)_j \lambda$$

$$- \frac{\partial \lambda}{\partial y}(H^{-1}p)_i x_j \quad \text{for all } i \text{ and } j, \tag{E2.17.5}$$

where the notations $(\cdot)_j$ and $(\cdot)_{ij}$ denote the jth and ijth components of the bracketed vector and matrix.

Next, recognize that under the additivity assumption, H and H^{-1} are diagonal matrices. Now use (E2.17.4) and (E2.17.5) to obtain (E2.17.1).

(d) What is the interpretation of the Frisch parameter, ω?

(e) Consider the utility function

$$V(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} V^i(x_i). \qquad \text{(E2.17.3)}$$

Would you expect restriction (E2.17.1) to be applicable? How about the interpretation of ω, would it still be the same as in the additive case?

(f) Can you complete the following scheme under the assumption that the consumer's preferences are additive?

$\alpha_1 = \frac{1}{3}$,	$\alpha_2 = \frac{1}{3}$,	$\alpha_3 = \frac{1}{3}$	
$e_{11} = -\frac{13}{48}$	$e_{12} = ?$	$e_{13} = ?$	$E_1 = \frac{1}{2}$
$e_{21} = ?$	$e_{22} = ?$	$e_{23} = ?$	$E_2 = \frac{1}{2}$
$e_{31} = ?$	$e_{32} = ?$	$e_{33} = ?$	$E_3 = ?$

Answer. (a) Under additivity we are assuming that the consumer behaves as if his marginal utility of good i is independent of his consumption of good j, $j \neq i$. This assumption is hard to justify for a very detailed study. The consumer's marginal utility for cotton shirts is likely to depend on both the number of cotton and synthetic shirts which he has. On the other hand, it may be acceptable to assume that the marginal utility for 'food' is independent of quantities of 'clothing'. Very intuitively, the additivity assumption rules out the possibility of 'special' substitution effects. It is not applicable when we have a situation in which i and j are extremely close substitutes, but i and k are only weakly substitutable.

(b) From (E2.17.1) we have

$$\alpha_i(e_{ij} + E_i\alpha_j) = -\frac{E_i\alpha_j E_j\alpha_i}{\omega}, \quad i \neq j,$$

and

$$\alpha_j(e_{ji} + E_j\alpha_i) = -\frac{E_j\alpha_i E_i\alpha_j}{\omega}, \quad i \neq j.$$

Hence

$$\alpha_i(e_{ij} + E_i\alpha_j) = \alpha_j(e_{ji} + E_j\alpha_i), \quad \text{for } i \neq j,$$

and we have shown that (E2.17.1) implies the symmetry condition (E2.6.2). On the other hand, the symmetry condition certainly does not imply (E2.17.1). In E2.7 you generated a scheme of elasticities which satisfied (E2.6.2). A little arithmetic will show that they do not satisfy (E2.17.1). For example, you will find that

$$\frac{\alpha_1(e_{12}+E_1\alpha_2)}{\alpha_1(e_{13}+E_1\alpha_3)} \neq \frac{E_1E_2\alpha_1\alpha_2}{E_1E_3\alpha_1\alpha_3} .$$

(c) One way to prove (E2.17.2) is by multiplying the bordered Hessian by our proposed inverse and checking that the result is the identity matrix. A more instructive method is as follows: let

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A & b \\ b' & \tau \end{bmatrix} \tag{i}$$

where A is an $n \times n$ matrix, b is an $n \times 1$ vector, and τ is a scalar.

We have written (i) so that the partitioning in the inverse is the same as that in the initial bordered Hessian. We have also used the symmetry of the bordered Hessian. Notice that we have taken account of the fact that the $(n + 1)$th row and $(n + 1)$th column of the inverse are the same.

From (i) we have

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} A & b \\ b' & \tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}. \tag{ii}$$

Our objective now is to manipulate (ii) so as to express A, b and τ in terms of H and p. By multiplying out the left-hand side of (ii) we find that

$$HA + pb' = I, \tag{iii}$$

$$Hb + p\tau = 0, \tag{iv}$$

$$p'A = 0, \tag{v}$$

$$p'b = 1. \tag{vi}$$

We assume that H has an inverse so that (iii) can be solved for A, i.e.

$$A = H^{-1}(I - pb'), \tag{vii}$$

and (iv) can be solved for b, i.e.

$$b = -H^{-1}p\tau. \tag{viii}$$

Multiplying (viii) by p' and using (vi), we obtain

$$\tau = -1/(p'H^{-1}p).$$

Substitution back into (viii) yields

$$b = \frac{1}{p'H^{-1}p} H^{-1}p,$$

and finally, substitution into (vii) gives

$$A = H^{-1} - \frac{1}{p'H^{-1}p} H^{-1}pp'H^{-1}.$$

(Since H and H^{-1} are symmetric, $(H^{-1}p)' = p'H^{-1}$.) Returning to (i) we see that

$$\begin{bmatrix} H & p \\ p' & 0 \end{bmatrix}^{-1} = \left(\frac{1}{p'H^{-1}p}\right) \left[\begin{array}{c|c} (p'H^{-1}p)H^{-1} - H^{-1}pp'H^{-1} & H^{-1}p \\ \hline p'H^{-1} & -1 \end{array}\right].$$

At this stage we can rewrite the matrix equation (E2.6.1) as

$$\begin{bmatrix} \mathrm{d}x \\ -\mathrm{d}\lambda \end{bmatrix} = \frac{1}{p'H^{-1}p} \left[\begin{array}{c|c} (p'H^{-1}p)H^{-1} - H^{-1}pp'H^{-1} & H^{-1}p \\ \hline p'H^{-1} & -1 \end{array}\right] \begin{bmatrix} \lambda \mathrm{d}p \\ \mathrm{d}y - (\mathrm{d}p)'x \end{bmatrix}$$

If we set $\mathrm{d}y = 1$ and $\mathrm{d}p = 0$, we find that

$$-\mathrm{d}\lambda = -\frac{1}{p'H^{-1}p}$$

and

$$\mathrm{d}x_i = \frac{1}{p'H^{-1}p} (H^{-1}p)_i.$$

Hence

$$\frac{\partial\lambda}{\partial y} = \frac{1}{p'H^{-1}p}$$

and

$$\frac{\partial x_i}{\partial y} = (H^{-1}p)_i \frac{\partial\lambda}{\partial y}.$$

Similarly, if we set $\mathrm{d}p_j = 1$ with $\mathrm{d}y$ and $\mathrm{d}p_k = 0$, for all $k \neq j$, we can derive (E2.17.5). Notice that the i, jth element of the matrix $H^{-1}pp'H^{-1}$ is the ith element of the vector $H^{-1}p$ multiplied by the jth element of the vector $p'H^{-1}$, i.e.

$$(H^{-1}pp'H^{-1})_{ij} = (H^{-1}p)_i\,(H^{-1}p)_j.$$

The time has now arrived for us to use the additivity assumption. Additivity implies that

$$\frac{\partial^2 U}{\partial x_i\,\partial x_j} = 0, \quad \text{for all} \quad i \neq j.$$

Hence the Hessian, H, and its inverse, are diagonal, and for $i \neq j$, the first term on the right-hand side of (E2.17.5) is zero. Thus (E2.17.5) reduces to

$$\frac{\partial x_i}{\partial p_j} = -\frac{\partial \lambda}{\partial y}\,(H^{-1}p)_i(H^{-1}p)_j\lambda - \frac{\partial \lambda}{\partial y}\,(H^{-1}p)_i x_j, \quad i \neq j. \tag{ix}$$

From (E2.17.4), we have

$$(H^{-1}p)_k = \left(\frac{\partial x_k}{\partial y}\right)\Big/\left(\frac{\partial \lambda}{\partial y}\right), \quad \text{for all} \quad k,$$

and by substitution into (ix) we obtain

$$\frac{\partial x_i}{\partial p_j} = -\frac{\partial x_i}{\partial y}\,\frac{\partial x_j}{\partial y}\left(\frac{\lambda}{\partial\lambda/\partial y}\right) - \frac{\partial x_i}{\partial y}\,x_j, \quad \text{for all} \quad i \neq j. \tag{x}$$

Eq. (x) is translated into a relationship between elasticities by multiplying and dividing by prices and quantities in the usual way:

$$\left[\frac{\partial x_i}{\partial p_j}\right]\frac{p_j}{x_i} = -\left[\frac{\partial x_i}{\partial y}\right]\frac{y}{x_i}\left[\frac{\partial x_j}{\partial y}\right]\frac{y}{x_j}\,\frac{p_j x_j}{y}\left[\frac{\lambda}{\partial\lambda/\partial y}\right]\frac{1}{y}$$

$$-\left[\frac{\partial x_i}{\partial y}\right]\frac{y}{x_i}\,\frac{[x_j]\,p_j}{y},$$

i.e.

$$e_{ij} = -\frac{E_i E_j \alpha_j}{\omega} - E_i\alpha_j \quad \text{for all} \quad i \neq j,$$

where

$$\omega = \frac{\partial\lambda}{\partial y}\,\frac{y}{\lambda}\,. \tag{xi}$$

On rearranging, we obtain (E2.17.1).

(d) From (xi) we see that ω can be interpreted as the elasticity of the marginal utility of income with respect to income, i.e. it shows the percentage effect on the marginal utility of income of a one percent increase in income.

(e) Utility function (E2.17.3) becomes additive under a positive monotonic transformation. Note that

$$\ln V = \sum_i \ln V^i.$$

Since the preferences described by the utility function, U, where

$$U = \ln V = \sum_i \ln V^i$$

are precisely the same as those described by the initial utility function, V, the demand responses to price and income movements will be unchanged by the replacement of V by U. (E2.17.1) is valid when the utility function is U. It will also be valid when the utility function is V.

With the multiplicative utility function V, ω cannot be interpreted as the elasticity of the marginal utility of income. However, that interpretation is valid when the utility function is additive. Hence

$$\omega = \frac{\partial(\partial \ln V/\partial y)}{\partial y} \frac{y}{\partial \ln V/\partial y},$$

i.e.

$$\omega = \frac{\partial\left(\dfrac{1}{V}\dfrac{\partial V}{\partial y}\right)}{\partial y} \frac{y}{\dfrac{1}{V}\dfrac{\partial V}{\partial y}}$$

$$= -\frac{\partial V}{\partial y}\frac{y}{V} + \frac{\partial(\partial V/\partial y)}{\partial y}\frac{y}{\partial V/\partial y}.$$

We can conclude that with the multiplicative utility function (E2.17.3), ω can be interpreted as the difference between the elasticity of *marginal* utility with respect to income and the elasticity of *total* utility with respect to income.

To summarize, if there exists F, a monotonically increasing function, such that $F(V)$ is additive, where V is the utility function, then restriction (E2.17.1) is applicable. However, the standard interpretation (but not the numerical value) of ω depends on the utility function being additive.

(f) From the Engel aggregation, (E2.1.3), we find that

$$E_3 = \frac{1 - \alpha_1 E_1 - \alpha_2 E_2}{\alpha_3} = 2.$$

Next, we use the homogeneity restriction (E2.2.1) to write

$$e_{11} + e_{12} + e_{13} = -E_1.$$

Under additivity, this last equation may be rewritten as

$$e_{11} - E_1\alpha_2\left(1 + \frac{E_2}{\omega}\right) - E_1\alpha_3\left(1 + \frac{E_3}{\omega}\right) = -E_1.$$

Now, by using the values for the α's, E's and e_{11} as shown in the table, we find that

$$\omega = -4.$$

From here we can use (E2.17.1) to generate the e_{ij} for all $i \neq j$ and the homogeneity restriction to fill in the diagonal terms e_{22} and e_{33}. The completed table is as follows:

$\alpha_1 = \frac{1}{3}$,	$\alpha_2 = \frac{1}{3}$,	$\alpha_3 = \frac{1}{3}$	
$e_{11} = -\frac{13}{48}$	$e_{12} = -\frac{7}{48}$	$e_{13} = -\frac{4}{48}$	$E_1 = \frac{1}{2}$
$e_{21} = -\frac{7}{48}$	$e_{22} = -\frac{13}{48}$	$e_{23} = -\frac{4}{48}$	$E_2 = \frac{1}{2}$
$e_{31} = -\frac{28}{48}$	$e_{32} = -\frac{28}{48}$	$e_{33} = -\frac{40}{48}$	$E_3 = 2$

On the basis of knowing the budget shares and two of the expenditure elasticities, plus one price elasticity, we were able to deduce the other nine elasticities. In general, we have $n^2 + n$ price and income elasticities. Under the assumption that the utility function is additive, we have

$$1 + n + (n(n-1) - 1) = n^2$$

restrictions: one restriction for the Engel aggregation, n restrictions for homogeneity and $n(n\text{-}1) - 1$ restrictions for additivity. Notice that (E2.17.1) determines the $n(n\text{-}1)$ cross-elasticities, but introduces one new elasticity ω. With $n = 3$, the additivity model provides $1 + 3 + 5 = 9$ restrictions applying to 12 elasticities. With $n = 20$, we have 400 restrictions applying to 420 elasticities.