

Chapter 3

Modelling Area Data

Abstract Exploratory spatial data analysis is often a preliminary step to more formal modelling approaches that seek to establish relationships between the observations of a variable and the observations of other variables, recorded for each areal unit. The focus in this chapter is on spatial regression models in a simple cross-sectional setting, leaving out of consideration the analysis of panel data. We, moreover, assume that the data concerned can be taken to be approximately normally distributed. This assumption is—to varying degrees—involved in most of the spatial regression techniques that we will consider. Note that the assumption of normality is not tenable if the variable of interest is a count or a proportion. In these cases we would expect models for such data to involve probability distributions such as the Poisson or binomial. The chapter consists of five sections, starting with a treatment of the specification of spatial dependence in a regression model. Next, specification tests are considered to detect the presence of spatial dependence. This is followed by a review of the spatial Durbin model (SDM) that nests many of the models widely used in the literature, and by a discussion of spatial regression model estimation based on the maximum likelihood (ML) principle. The chapter closes with some remarks on model parameter interpretation, an issue that had been largely neglected so far. Readers interested in implementing the models, methods and techniques discussed in this chapter find useful MATLAB code which is publicly available at spatial-econometrics.com, LeSage's spatial econometrics toolbox (downloadable from <http://www.spatial-econometrics.com/>), see Liu and LeSage (2010) *Journal of Geographical Systems* 12(1):69–87 for a brief description. Another useful open software is the `spdep` package of the R project (downloadable from <http://cran.r-project.org>).

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3.1 Spatial Regression Models

Starting point is the linear regression model, where for each observation (area) i , with $i = 1, \dots, n$, the following relationship holds

$$y_i = \sum_{q=1}^Q X_{iq} \beta_q + \varepsilon_i \quad (3.1)$$

where y_i is an observation on the dependent variable, X_{iq} is an observation on an explanatory variable, with $q = 1, \dots, Q$ (including a constant, or one), β_q the matching regression coefficient, and ε_i the error term.

In the classical regression specification, the error terms have mean zero, that is, $E[\varepsilon_i] = 0$ for all i , and they are identically and independently distributed (*iid*). Hence, their variance is constant, $\text{var}[\varepsilon_i] = \sigma^2$ for all i , and they are uncorrelated, $E[\varepsilon_i \varepsilon_j] = E[\varepsilon_i]E[\varepsilon_j] = 0$ for $i \neq j$.

In matrix notation this regression model may be written as

$$y = X\beta + \varepsilon \quad (3.2)$$

where the n observations on the dependent variable are stacked in an n -by-1 vector y , the observations on the explanatory variables in an n -by- Q matrix X with the associated Q -by-1 parameter vector β , and the random error terms in an n -by-1 vector ε . $E[\varepsilon] = 0$ where 0 is an n -by-1 vector of zeros, and $E[\varepsilon \varepsilon'] = \sigma^2 I$ with I denoting the n -by- n identity matrix.

The assumption of independent observations greatly simplifies the model, but in the context of area data this simplification is very unlikely to be appropriate, because of the possibility of spatial dependence between the error terms. If the regressors, residuals or the dependent variable are spatially dependent, the model suffers from a misspecification problem and the results of the model are biased or inconsistent.

Spatial dependence reflects a situation where values observed in one areal unit, depend on the values of neighbouring observations at near-by areas. Spatial dependence may be introduced into a model of type (3.2) in two major ways: one is referred to as spatial lag dependence, and the other as spatial error dependence (Anselin 1988b). The former pertains to spatial correlation in the dependent variable, while the latter refers to the error term. Hence, it has become convenient to distinguish between spatial lag and spatial error model specifications.

Spatial dependence can also be introduced in the regressor variables, leading to so-called cross-regressive models (Florax and Folmer 1992), also termed spatially lagged X (or SLX) models (LeSage and Pace 2009). But—in contrast to the spatial lag and spatial error models—they do not require the application of special estimation procedures. Thus, they will not be further considered in this chapter.

Spatial lag models account for spatial correlation (dependence) in the dependent variable. Such specifications are typically motivated by theoretical arguments

that emphasise the importance of neighbourhood effects, or spatial externalities that cross the borders of the areal units and show up in the dependent variable. This kind of spatial autocorrelation is substantive on the ground that it has a meaningful interpretation.

In contrast, spatial error models account for spatial dependence in the error term. Spatial error dependence may arise, for example, from unobservable latent variables that are spatially correlated. It may also arise from area boundaries that do not accurately reflect neighbourhoods which give rise to the variables collected for the analysis. Spatial autocorrelation arising for these reasons is considered to be nuisance.

Spatial Lag Models Spatial lag models are extensions of regression models of type (3.1). They allow observations of the dependent variable y in area i ($i = 1, \dots, n$) to depend on observations in neighbouring areas $j \neq i$. The basic spatial lag model, the so-called first order *spatial autoregressive* (SAR) model, takes the form

$$y_i = \rho \sum_{j=1}^n W_{ij} y_j + \sum_{q=1}^Q X_{iq} \beta_q + \varepsilon_i \quad (3.3)$$

where the error term, ε_i , is *iid*. W_{ij} is the (i, j) th element of the n -by- n spatial weights matrix W (see Sect. 2.2). Recall that W has non-zero elements W_{ij} in each row i for those columns j that are neighbours of area i . By convention, $W_{ii} = 0$ for all i . All these values are exogenous. We assume that W is row-stochastic so that the matrix W has a principal eigenvalue of one. The term row-stochastic refers to a non-negative matrix having row sums normalised so they equal one.

The scalar ρ in Eq. (3.3) is a parameter (to be estimated) that will determine the strength of the spatial autoregressive relation between y_i and $\sum_j W_{ij} y_j$, a linear combination of spatially related observations based on non-zero elements in the i th row of W . The domain of ρ is defined by the interval $(w_{\min}^{-1}, w_{\max}^{-1})$, where w_{\min} and w_{\max} represent the minimum and maximum eigenvalues of the matrix W . For the case of a row-normalised weights matrix, $-1 \leq w_{\min} < 0$, $w_{\max} = 1$ so that ρ ranges from negative values to unity. In cases where positive spatial dependence is almost certain, restriction of ρ to the $[0, 1)$ interval simplifies computation. It should be clear that if $\rho = 0$, we have a conventional regression model of type (3.1) so that interest focuses on the statistical significance of the coefficient estimate for ρ .

In matrix notation, model (3.3) may be written as

$$y = \rho W y + X \beta + \varepsilon. \quad (3.4)$$

With a row-standardised spatial weights matrix W (that is, the weights are standardised such that $\sum_j W_{ij} = 1$ for all i), this amounts to including the average of the neighbours as an additional variable into the regression specification. This variable, Wy , is referred to as a spatially lagged dependent variable. For example, in a model for growth rates of European regions, this would add the average of the

growth rates in the neighbouring locations as an explanatory variable. The model given by Eq. (3.4) is a structural model. Its reduced form, that is, the solution of the model for y is

$$y = (I - \rho W)^{-1}(X\beta + \varepsilon) \quad (3.5)$$

so that the expected value of y is

$$E[y] = (I - \rho W)^{-1}X\beta \quad (3.6)$$

since the errors all have mean zero. The inverse matrix term is called *spatial multiplier*, and indicates that the expected value of each observation y_i will depend on a linear combination of X -values taken by neighbouring observations, scaled by the dependence parameter ρ .

Spatial Error Models Another form of spatial dependence occurs when the dependence works through the error process, in that the errors from different areas may display spatial covariance. The most common specification is a spatial autoregressive process of first order, as given by

$$\varepsilon_i = \lambda \sum_{j=1}^n W_{ij}\varepsilon_j + u_i \quad (3.7)$$

where λ is the autoregressive parameter, and u_i a random error term, typically assumed to be *iid*. In matrix notation Eq. (3.7) may be reformulated as

$$\varepsilon = \lambda W\varepsilon + u. \quad (3.8)$$

Assuming $|\lambda| < 1$ and solving Eq. (3.8) for ε yields

$$\varepsilon = (I - \lambda W)^{-1}u. \quad (3.9)$$

Inserting Eq. (3.9) into the standard regression model (3.2) yields the spatial error model

$$y = X\beta + (I - \lambda W)^{-1}u \quad (3.10)$$

with $E[uu'] = \sigma^2 I$ so that the complete error variance–covariance matrix follows as

$$E[\varepsilon\varepsilon'] = \sigma^2(I - \lambda W)^{-1}(I - \lambda W')^{-1}. \quad (3.11)$$

The spatial error model (SEM) may be viewed as a combination of a standard regression model with a spatial autoregressive model in the error term ε , and hence has an expectation equal to that of the standard regression model. In large samples, point estimates for the parameters β from the SEM model and conventional regression will be the same, but in small samples there may be an efficiency gain from correctly modelling spatial dependence in the error terms. Note that in contrast spatial lag models that contain spatial lag terms Wy on the right-hand side of the equation generate expectations that differ from those of the standard regression model.

Higher Order Models In addition to the basic spatial lag and spatial error models described above, higher order models can be specified as well, by including two or more weight matrices. Using multiple weight matrices provides a straightforward generalisation of the SAR and SEM models. For example, Anselin (1988b, pp. 34–36) uses two spatial weights matrices W_1 and W_2 to combine the basic spatial lag and error models so that

$$y = \rho W_1 y + X\beta + \varepsilon \quad (3.12)$$

$$\varepsilon = \lambda W_2 \varepsilon + u \quad (3.13)$$

$$u \sim \mathcal{N}(0, \sigma_u^2 I) \quad (3.14)$$

where W_1 and W_2 represent n -by- n non-negative spatial weights matrices (not necessarily distinct from each other) with zeros on the main diagonal. The parameters to be estimated are β, ρ, λ and σ_u^2 . Setting the parameter $\rho = 0$ eliminates the spatially lagged variable $W_1 y$, generating the basic spatial error model given by Eq. (3.10). The case where $\lambda = 0$ eliminates the spatially lagged disturbance term yields the basic spatial lag model given by Eq. (3.4).

3.2 Tests for Spatial Dependence

The standard approach towards detecting the presence of spatial dependence in a regression model is to apply diagnostic tests. The best known test statistic against spatial autocorrelation is Moran's I statistic for spatial autocorrelation applied to the regression residuals (see Cliff and Ord 1972, 1973, see also Sect. 2.3):

$$I = \frac{n}{W_0} \frac{e' W e}{e' e} \quad (3.15)$$

$$W_0 = \sum_{i=1}^n \sum_{j \neq i}^n W_{ij} \quad (3.16)$$

where e is an n -by-1 vector of OLS residuals $y - X\hat{\beta}$, $e' e$ is the sum of squared residuals, and W_0 , equal to the sum of W_{ij} over i and j , is a normalising factor. Note that the correcting factor n/W_0 is not needed if the spatial weights matrix W is row-standardised. In practice, inference—by means of Moran's I test—is based on a normal approximation, using a standardised value, obtained by subtracting the mean under the null and dividing by the square root of the variance.

As already pointed out in Sect. 2.3, care needs to be taken when applying this formal test of spatial dependence to residuals. The problem arises because if Q regression coefficients have been estimated, then the observed residuals are automatically subject to Q linear constraints. That is, the observed residuals will be correlated to some extent, and hence the testing procedure for Moran's I will not

be valid. If $Q \ll n$, however, then it might be justified in ignoring this. If not, then strictly one should use adjustments to the mean and variance of the approximate sampling distribution of I .

An alternative, more focused test for spatial error dependence is based on the Lagrange multiplier (LM) principle, suggested by Burridge (1980). It is similar in expression to Moran's I and is also computed from the OLS residuals. But a normalisation factor in terms of matrix traces is needed to achieve an asymptotic chi-square distribution (with one degree of freedom) under the null hypothesis of no spatial dependence ($H_0 : \lambda = 0$). The LM error statistic is given by

$$LM(\text{error}) = \left(\frac{e'We}{e'e n^{-1}} \right)^2 \frac{1}{tr[W'W + W^2]} \quad (3.17)$$

where tr stands for the trace operator (the sum of the diagonal elements of a matrix), and $(e'e n^{-1})$ represents the error variance. Except for the scaling factor $tr[W'W + W^2]^{-1}$, this statistic is essentially the square of Moran's I .

A test for substantive spatial dependence (that is, an omitted spatial lag) can also be based on the Lagrange multiplier principle (see Anselin 1988b). Its form is slightly more complex, but again requires only the results of an OLS regression. The test takes the form

$$LM(\text{lag}) = \left(\frac{e'Wy}{e'e n^{-1}} \right)^2 \frac{1}{H} \quad (3.18)$$

with

$$H = \{(WX\hat{\beta})'[I - X(X'X)^{-1}X'](WX\hat{\beta})\hat{\sigma}^{-2}\} + tr(W'W + W^2) \quad (3.19)$$

where $\hat{\beta}$ and $\hat{\sigma}^2$ denote OLS estimates, Wy is the spatial lag and $WX\hat{\beta}$ is a spatial lag for the predicted values $X\hat{\beta}$, and $[I - X(X'X)^{-1}X']$ is a familiar projection matrix. The $LM(\text{lag})$ test is also distributed as chi-square with one degree of freedom under the null hypothesis of no spatial dependence ($H_0 : \rho = 0$).

Specification Search For the simple case of choosing between a spatial lag or spatial error alternative, there is considerable evidence that the proper alternative is most likely the one with the largest significant LM test statistic value (Anselin and Rey 1991). This was later refined in light of the robust form of the two LM statistics in Anselin et al. (1996) accounting for the fact that in the presence of spatial lag (error) dependence, the LM test against error (lag) dependence becomes biased.

Florax and Folmer (1992) suggest a sequential testing procedure to discern whether a model based on the restrictions $\rho = 0$ versus $\lambda = 0$, versus both ρ and λ different from zero should be selected. Of course, this approach complicates inference concerning the parameters of the final model specification due to the pre-test issue.

Florax et al. (2003) consider Hendry's "general to specific approach" to model specification versus a forward stepwise strategy. While the "general to specific" approach tests sequential restrictions placed on the most general model

(3.12)–(3.14) that includes both spatial lag and spatial error dependence, the stepwise strategy considers sequential expansions of the model. Starting with regression model (3.2), expansion of the model proceeds to add spatial lag terms, conditional upon the results of misspecification tests. They conclude that the Hendry approach is inferior in its ability to detect that true data generating process.

3.3 The Spatial Durbin Model

The spatial Durbin model (SDM) is the SAR model (3.4) augmented by spatially lagged explanatory variables

$$y = \rho W y + X \beta + W \bar{X} \gamma + \varepsilon \quad (3.20)$$

where \bar{X} is the n -by- $(Q-1)$ non-constant explanatory variable matrix. The model may be rewritten in reduced form as

$$y = (I - \rho W)^{-1} (X \beta + W \bar{X} \gamma + \varepsilon) \quad (3.21)$$

with

$$\varepsilon = \mathcal{N}(0, \sigma^2 I) \quad (3.22)$$

where γ is a $(Q-1)$ -by-1 vector of parameters that measure the marginal impact of the explanatory variables from neighbouring observations (areas) on the dependent variable y . Multiplying \bar{X} by W (that is, $W \bar{X}$) produces spatially lagged explanatory variables that reflect an average of neighbouring observations. If W is sparse (having a large proportion of zeros), operations such as $W \bar{X}$ require little time.

By defining $Z = [X \ W \bar{X}]$ and $\delta = [\beta \ \gamma]'$ this model can be written as a SAR model leading to

$$y = \rho W y + Z \delta + \varepsilon \quad (3.23)$$

or

$$y = (I - \rho W)^{-1} Z \delta + (I - \rho W)^{-1} \varepsilon. \quad (3.24)$$

One motivation for use of the SDM model rests on the plausibility of a conjunction of two circumstances that seem likely to arise in applied spatial regression modelling of area data samples. One of these is spatial dependence in the disturbances of an OLS regression model. The second circumstance is the existence of an omitted explanatory variable that exhibits non-zero covariance with a variable included in the model, and omitted variables are likely when dealing with areal data samples (LeSage and Fischer 2008).

In addition, the spatial Durbin model occupies an interesting position in the field of spatial regression analysis because it nests many of the models widely used in the literature (see LeSage and Pace 2009):

- (i) imposing the restriction $\gamma = 0$ leads to the spatial autoregressive model (3.4) that includes a spatial lag of the dependent variable, but excludes the influence of the spatially lagged explanatory variables,
- (ii) the so-called common factor parameter restriction $\gamma = -\rho \bar{\beta}$ yields the spatial error regression model specification (3.10) which assumes that externalities across areas are mostly a nuisance spatial dependence problem caused by the spatial transmission of random shocks (Note that $\bar{\beta}$ denotes the $(Q-1)$ -by-1 vector of parameters that measure the marginal impact of the non-constant explanatory variables on the dependent variable. $\beta = (\beta_0, \bar{\beta})'$ where β_0 is the constant term parameter),
- (iii) the restriction $\rho = 0$ results in a least squares spatially lagged X regression model that assumes independence between observations of the dependent variable, but includes characteristics from neighbouring areas, in the form of spatially lagged explanatory variables,
- (iv) finally, imposing the restriction $\rho = 0$ and $\gamma = 0$ yields the standard least squares regression model given by Eq. (3.2).

Hence, the SDM model suggests a general-to-simple model selection rule. Testing, whether the restrictions hold or not, implies not much effort. Of particular importance are common factor tests that discriminate between the unrestricted SDM and the SEM specifications, or in other words between substantive and residual dependence in the analysis. The likelihood ratio test proposed by Burridge (1980) is the most popular test in this context (see LeSage and Pace (2009) for details, Mur and Angulo (2006) for alternative tests and a comparison based on Monte Carlo evidence).

Finally, it should be noted that the spatial Durbin model (3.20) can be generalised to

$$y = \rho W_1 y + X\beta + W_1 \bar{X}\gamma + \varepsilon \quad (3.25)$$

$$\varepsilon = \lambda W_2 \varepsilon + u \quad (3.26)$$

$$u \sim \mathcal{N}(0, \sigma_u^2 I), \quad (3.27)$$

where the n -by- n spatial weights matrices W_1 and W_2 can be the same or distinct. For details on this model generalisation see LeSage and Pace (2009, pp. 52–54).

3.4 Estimation of Spatial Regression Models

Estimation of spatial regression models is typically carried out by means of a maximum likelihood (ML) approach, in which the probability of the joint distribution (likelihood) of all observations is maximised with respect to a number of relevant parameters. Maximum likelihood estimation has desirable asymptotic theoretical properties such as consistency, efficiency and asymptotic normality,

and is also thought to be robust for small departures from the normality assumption (LeSage and Pace 2004, pp. 10–11).

The estimation problems associated with spatial regression models are different for the spatial lag and the spatial error cases. We start the discussion by focusing on the SAR (and SDM) model presented in Eq. (3.23).

Given $\varepsilon \sim \mathcal{N}(0, \sigma^2 I)$, the log (more precisely the logarithm naturalis) of the likelihood for the SAR model given by Eq. (3.23) takes the form in Eq. (3.28) (Anselin 1988b, p. 63)

$$\begin{aligned} \ln L(\rho, \delta, \sigma^2) &= -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 + \ln |A| \\ &\quad - \frac{1}{2\sigma^2} (Ay - Z\delta)' (Ay - Z\delta) \end{aligned} \quad (3.28)$$

where n is the number of observations, $|\cdot|$ stands for the determinant of a matrix, and for notational simplicity, the expression $I - \rho W$ is replaced by A . The parameters with respect to which this likelihood has to be maximised are ρ , δ and σ^2 .

The minimisation of the last term in Eq. (3.28) corresponds to ordinary least squares (OLS), but since this ignores the log-Jacobian term $\ln |I - \rho W|$, OLS is not a consistent estimator in this model. There is no satisfactory two-step procedure and estimators for the parameters have to be obtained from an explicit maximisation of the likelihood (Anselin 2003b).

But it turns out that the estimates for the regressive coefficients δ , conditional upon the value for ρ , can be found as

$$\delta = \delta_O - \rho \delta_L \quad (3.29)$$

where δ_O and δ_L are OLS regression coefficients in a regression of Z on y and Wy , respectively. In a similar way the error variance σ^2 can be estimated as

$$\sigma^2 = (e_O - \rho e_L)' (e_O - \rho e_L) \frac{1}{n} \quad (3.30)$$

where e_O and e_L are the residual vectors in the regressions for δ_O and δ_L . That is, $e_O = y - Z\delta_O$ and $e_L = Wy - Z\delta_L$, where $\delta_O = (Z'Z)^{-1}Z'y$ and $\delta_L = (Z'Z)^{-1}Z'Wy$.

Substitution of (3.29) and (3.30) into the log-likelihood function (3.28) gives the scalar concentrated log-likelihood function value

$$\ln L_{con}(\rho) = \kappa + \ln |I - \rho W| - \frac{n}{2} \ln [(e_O - \rho e_L)' (e_O - \rho e_L)] \quad (3.31)$$

where κ is a constant that does not depend on ρ . The motivation for optimising the concentrated log-likelihood is that this simplifies the optimising problem by reducing a multivariate optimisation problem to a univariate one. Maximising the concentrated log-likelihood function with respect to ρ yields ρ^* that is equal to the maximum likelihood estimate ($\hat{\rho}_{ML} = \rho^*$). Note that it is well-known that maximum likelihood often has a downward bias in estimation of ρ in small samples.

The computationally difficult aspect of this optimisation problem for models with a large number of observations is the need to compute the log-determinant of the n -by- n matrix $(I - \rho W)$. In response to this computational challenge there are at least two strategies. *First*, the use of alternative estimators can solve this problem. Examples include the instrumental variable (IV) approach (Anselin 1988b, pp. 81–90) and the instrumental variable (IV)/generalised moments (GM) approach (Kelejian and Prucha 1998, 1999). These alternative estimation methods, however, suffer from several drawbacks. One is that they can produce ρ -estimates that fall outside the interval defined by the eigenvalue bounds arising from the spatial weights matrix W . Moreover, inferential procedures for these methods can be sensitive to implementation issues such as the interaction between the choice of instruments and model specification which are not always obvious to the practitioner (LeSage and Pace 2010).

A *second* strategy is to directly attack the computational difficulties confronting ML estimation. The Taylor series approach of Martin (1993), the eigenvalue based approach of Griffith and Sone (1995), the direct sparse matrix approach of Pace and Barry (1997), the characteristic polynomial approach of Smirnov and Anselin (2001), and the sampling approach of Pace and LeSage (2009) are examples of this strategy. A review of most of the approximations to the log-determinant can be found in LeSage and Pace (LeSage and Pace 2009, Chap. 4). Improvements in computing technology in combination with these approaches suggest that very large problems can be handled today, using the ML estimation approach.

Inference regarding parameters for the models is frequently based on estimates of the variance–covariance matrix. In problems where the sample size is small, an asymptotic variance matrix based on the Fisher information matrix for parameters $\eta = (\rho, \delta, \sigma^2)$ can be used to provide measures of dispersion for these parameters. Anselin (1988b) provides the analytical expressions needed to construct this information matrix, but evaluating these expressions may be computationally difficult when dealing with large scale problems involving thousands of observations (LeSage and Pace 2004, p. 13).

Let us turn next to the spatial error model presented in Eq. (3.10) that represents another member of the family of regression models that can be derived from Eqs. (3.12)–(3.14). Assuming normality for the error terms, and using the concept of a Jacobian for this model as well, the log-likelihood for the SEM model can be obtained as

$$\begin{aligned} \ln L(\lambda, \beta, \sigma^2) = & -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 + \ln |I - \lambda W| \\ & - \frac{1}{2\sigma^2} (y - X\beta)' (I - \lambda W)' (I - \lambda W) (y - X\beta). \end{aligned} \quad (3.32)$$

A closer inspection of the last term in Eq. (3.32) reveals that—conditional upon a given λ —a maximisation of the log-likelihood is equivalent to the minimisation of the sum of squared residuals in a regression of a spatially filtered dependent variable $y^* = y - \lambda Wy$ on a set of spatially filtered explanatory

variables $X^* = X - \lambda WX$. The first order conditions for $\hat{\beta}_{ML}$ indeed generate the familiar generalised least squares estimator (Anselin 2003b)

$$\hat{\beta}_{ML} = [X'(I - \lambda W)'(I - \lambda W)X]^{-1}X'(I - \lambda W)'(I - \lambda W)y \quad (3.33)$$

and, similarly, the ML estimator for σ^2 as

$$\hat{\sigma}_{ML}^2 = (e - \lambda We)'(e - \lambda We) \frac{1}{n} \quad (3.34)$$

where $e = y - X\hat{\beta}_{ML}$. A consistent estimate for λ cannot be obtained from a simple auxiliary regression, but the first order conditions must be solved explicitly by numerical means. For technical details, see Anselin (1988b, Chap. 6), or LeSage and Pace (2009, Chap. 3). As for the spatial lag model, asymptotic inference can be based on the inverse of the information matrix (see Anselin 1988b, Chap. 6, for details).

3.5 Model Parameter Interpretation

Simultaneous feedback is a feature of spatial regression models that comes from dependence relations embodied in the spatial lag term Wy . This leads to feedback effects from changes in explanatory variables in an area that neighbours i , say area j , that will impact the dependent variable for observation (area) i . Consequently, interpretation of parameters of spatial regression models that contain a spatial lag Wy becomes more complicated (see, for example, Kim et al. 2003; Anselin and LeGallo 2006; Kelejian et al. 2006; LeSage and Fischer 2008).

To see how these feedback effects work, we follow LeSage and Pace (2010) and consider the data generating process associated with the spatial lag model, shown in Eq. (3.35) to which we—assuming that ρ in absolute value is less than 1 and W is row-stochastic—have applied the well known infinite series expansion in Eq. (3.36) to express the inverse of $(I - \rho W)$

$$y = (I - \rho W)^{-1}X\beta + (I - \rho W)^{-1}\varepsilon \quad (3.35)$$

$$(I - \rho W)^{-1} = I + \rho W + \rho^2 W^2 + \rho^3 W^3 + \dots \quad (3.36)$$

$$\begin{aligned} y = & X\beta + \rho WX\beta + \rho^2 W^2 X\beta + \rho^3 W^3 X\beta + \dots \\ & + \varepsilon + \rho W\varepsilon + \rho^2 W^2 \varepsilon + \rho^3 W^3 \varepsilon + \dots \end{aligned} \quad (3.37)$$

The model statement in Eq. (3.37) can be interpreted as indicating that the expected value of each observation y_i will depend on the mean plus a linear combination of values taken by neighbouring observations (areal units), scaled by the dependence parameters $\rho, \rho^2, \rho^3, \dots$

Consider the powers of the row-stochastic spatial weights matrix W (that is, W^2, W^3, \dots) which appear in Eq. (3.37) where we assume that the rows of W are constructed to represent *first order* contiguous neighbours. Then the matrix W^2 will reflect *second order* contiguous neighbours, those that are neighbours to the first order neighbours. Since the neighbour of the neighbour (second order neighbour) to an observation i includes observation i itself, W^2 has positive elements on the main diagonal, when each observation has at least one neighbour. That is, higher order spatial lags can lead to a connectivity relation for an observation i such that $W^2X\beta$ and $W^2\varepsilon$ will extract observations from the vectors $X\beta$ and ε that point back to the observation i itself. This is in contrast to the conventional independence relation in ordinary least squares regressions where the Gauss-Markov assumptions rule out dependence of ε_i on other observations ($j \neq i$), by assuming zero covariance between observations i and j in the data generating process (LeSage and Pace 2010).

In standard least squares regression models of type (3.2) where the dependent variable vector contains *independent* observations, changes in observation i on the q th (non-constant) explanatory variable, which we denote by \bar{X}_{iq} , only influence observation y_i , so that the parameters have a straightforward interpretation as partial derivatives of the dependent variable with respect to the explanatory variable

$$\frac{\partial y_i}{\partial \bar{X}_{jq}} = \begin{cases} \beta_q & \text{for } i = j \text{ and } q = 1, \dots, Q-1 \\ 0 & \text{for } j \neq i \text{ and } q = 1, \dots, Q-1. \end{cases} \quad (3.38)$$

The SAR model (and the spatial Durbin model) allows this type of change to influence y_i as well as other observations y_j with $j \neq i$. This type of impact arises due to the interdependence or connectivity between observations in the model. To see how this works, consider the spatial lag model expressed as shown in Eq. (3.39)

$$y = \sum_{q=1}^Q S_q(W) \bar{X}_q + V(W) \iota_n \beta_0 + V(W) \varepsilon \quad (3.39)$$

$$S_q(W) = V(W) (I \beta_q) \quad (3.40)$$

$$V(W) = (I - \rho W)^{-1} \quad (3.41)$$

where β_0 is the constant term parameter on ι_n , the n -by-1 vector of ones. Note that in the case of the SDM model $S_q(W) = V(W) (I \beta_q + W \gamma_q)$. For more details see LeSage and Pace (2009, 34 pp.).

To illustrate the role of $S_q(W)$, we rewrite the expansion of the data generating process in Eq. (3.39) as shown in Eq. (3.42)

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{q=1}^{Q-1} \begin{pmatrix} S_q(W)_{11} & S_q(W)_{12} & \dots & S_q(W)_{1n} \\ S_q(W)_{21} & S_q(W)_{22} & \dots & S_q(W)_{2n} \\ \vdots & \vdots & & \vdots \\ S_q(W)_{n1} & S_q(W)_{n2} & \dots & S_q(W)_{nn} \end{pmatrix} \begin{pmatrix} \bar{X}_{1q} \\ \bar{X}_{2q} \\ \vdots \\ \bar{X}_{nq} \end{pmatrix} + V(W)_{1n}\beta_0 + V(W)\varepsilon. \quad (3.42)$$

To make the role of $S_q(W)$ clear, consider the determination of a single dependent variable observation y_i

$$y_i = \sum_{q=1}^Q [S_q(W)_{i1}\bar{X}_{1q} + S_q(W)_{i2}\bar{X}_{2q} + \dots + S_q(W)_{in}\bar{X}_{nq}] + V(W)_{in}\beta_0 + V(W)_i\varepsilon \quad (3.43)$$

where $S_q(W)_{ij}$ denotes the (i, j) th element of the matrix $S_q(W)$, and $V(W)_i$ the i th row of $V(W)$. It follows from Eq. (3.43) that—unlike to the case of the independent regression model—the derivative of y_i with respect to \bar{X}_{jq} ($j \neq i$) is potentially non-zero, taking a value determined by the (i, j) th element of the matrix $S_q(W)$, see LeSage and Pace (2010):

$$\frac{\partial y_i}{\partial \bar{X}_{jq}} = S_q(W)_{ij}. \quad (3.44)$$

In contrast to the least squares case, the derivative of y_i with respect to \bar{X}_{iq} usually does not equal β_q , but results in an expression $S_q(W)_{ii}$ that measures the impact on the dependent observation i from a change in \bar{X}_{iq}

$$\frac{\partial y_i}{\partial \bar{X}_{iq}} = S_q(W)_{ii}. \quad (3.45)$$

Hence, a change to an explanatory variable in a single area (observation) can affect the dependent variable in other areas (observations). This is a logical consequence of the simultaneous spatial dependence structure in the spatial lag model. A change in the characteristics of neighbouring areal units can set in motion changes in the dependent variable that will impact the dependent variable in neighbouring areas. These impacts will diffuse through the system of areas.

Since the partial derivatives take the form of an n -by- n matrix and since there are $Q-1$ non-constant explanatory variables, this results in $(Q-1)n^2$ partial derivatives which provides an overwhelming amount of information. LeSage and Pace (2009, pp. 36–37) suggest summarising these partial derivatives. In particular, they propose averaging all the column or row sums of $S_q(W)$ to arrive at the *average total impact* or *effect*, averaging the main diagonal elements of this matrix to arrive at the *average direct impact* or *effect*, and averaging the off-diagonal

elements of $S_q(W)$ to arrive at the *average indirect impact* or *effect*. This latter summary measure reflects what are commonly thought of as *spatial spillovers*, or impacts falling on areas other than the own-area.

One applied illustration that uses these scalar summary impact estimates can be found in Fischer et al. (2009b). The application considers the direct, indirect and total impacts of changes in human capital on labour productivity levels in European regions. A number of other applications can be found in LeSage and Pace (2009) in a wide variety of application contexts.

For inference regarding the significance of these impacts, one needs to determine their empirical or theoretical distributions. Since the impacts reflect a non-linear combination of the parameters ρ and $\bar{\beta}$ in the case of the SAR model, working with the theoretical distribution is not very convenient. Given the model estimates as well as the associated variance–covariance matrix along with the knowledge that the ML estimates are (asymptotically) normally distributed, one can simulate the parameters ρ and $\bar{\beta}$ (and γ in the case of the SDM model). These empirically simulated magnitudes can be used in the expressions for the scalar summary measures to generate an empirical distribution of the scalar impact measures (LeSage and Pace 2009, 2010).

An illustration of a simulation approach to determining measures of dispersion for these scalar summary impact estimates can be found in Fischer et al. (2009b). Another illustration is given in LeSage and Fischer (2008) in the context of Bayesian model averaging methods.