# 12 Supply-Side Models, Linkages, and Important Coefficients

## 12.1 Supply Side Input–Output Models

### *12.1.1 The Early Interpretation*

In 1958 Ghosh presented an alternative input–output model based on the same set of base-year data that underpin the demand-driven model in earlier chapters, namely $\mathbf{Z}$, $\mathbf{f}$, and $\mathbf{v}$, from which $\mathbf{x}$ follows as $\mathbf{x} = \mathbf{Zi}+\mathbf{f}$ or as $\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}'$. In the demand-driven model, direct input coefficients are defined in $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, leading to $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{Lf}$. In this case the Leontief inverse relates sectoral gross outputs to the amount of final product (final demand) – that is, to a unit of product *leaving* the interindustry system at the end of the process. The alternative interpretation that Ghosh suggests relates sectoral gross production to the primary inputs – that is, to a unit of value *entering* the interindustry system at the beginning of the process.

This approach is made operational by essentially "rotating" or transposing our vertical (column) view of the model to a horizontal (row) one. Instead of dividing each *column* of $\mathbf{Z}$ by the gross output of the sector associated with that column, the suggestion is to divide each *row* of $\mathbf{Z}$ by the gross output of the sector associated with that row. We use $\mathbf{B}$ to denote the *direct-output coefficients* matrix that results.[1] For a two-sector example, this means

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} z_{11}/x_1 & z_{12}/x_1 \\ z_{21}/x_2 & z_{22}/x_2 \end{bmatrix} = \begin{bmatrix} 1/x_1 & 0 \\ 0 & 1/x_2 \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \hat{\mathbf{x}}^{-1}\mathbf{Z} \tag{12.1}$$

These $b_{ij}$ coefficients represent the distribution of sector $i$'s outputs across sectors $j$ that purchase interindustry inputs from $i$; these are frequently called *allocation* coefficients, as opposed to *technical* coefficients, $a_{ij}$. Using

$$\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}'$$

[1] Early presentations used $\overrightarrow{\mathbf{A}}$ for these coefficients and $\mathbf{A}\downarrow$ for the traditional demand-side coefficients, which we have denoted simply by $\mathbf{A}$. This served to make visually explicit the two points of view: $\mathbf{A}\downarrow$ resulting from uniform division of all elements in each column of $\mathbf{Z}$ by the associated column output, and $\overrightarrow{\mathbf{A}}$ resulting from division of all elements in each row of $\mathbf{Z}$ by the associated row output.



Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 543.
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where $\mathbf{v}' = [v_1, \ldots, v_n]$ – this is (2.29) in Chapter 2 – and

$$\mathbf{Z} = \hat{\mathbf{x}}\mathbf{B} \tag{12.2}$$

from (12.1), we have

$$\mathbf{x}' = \mathbf{i}'\hat{\mathbf{x}}\mathbf{B} + \mathbf{v}' = \mathbf{x}'\mathbf{B} + \mathbf{v}' \tag{12.3}$$

since $\mathbf{i}'\hat{\mathbf{x}} = \mathbf{x}'$. From this,

$$\mathbf{x}' = \mathbf{v}'(\mathbf{I} - \mathbf{B})^{-1} \tag{12.4}$$

Define

$$\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} \tag{12.5}$$

with elements $g_{ij}$. This has been called the *output inverse*, in contrast to the usual Leontief inverse, $\mathbf{L} = [l_{ij}] = (\mathbf{I} - \mathbf{A})^{-1}$ (the *input inverse*). Element $g_{ij}$ has been interpreted as measuring "the total value of production that comes about in sector $j$ per unit of primary input in sector $i$." (Augustinovics, 1970, p. 252.) Then, (12.4) is

$$\mathbf{x}' = \mathbf{v}'\mathbf{G} \tag{12.6}$$

In terms of *changes* in $\mathbf{v}$, we would find the associated output changes as

$$\Delta\mathbf{x}' = (\Delta\mathbf{v}')\mathbf{G} \tag{12.7}$$

As we have seen earlier with the Leontief price model (section 2.6), we can equally well transpose all elements so that the resulting vector of gross outputs is a column rather than a row. In that case, (12.3) will be

$$\mathbf{x} = \mathbf{B}'\mathbf{x} + \mathbf{v} \tag{12.8}$$

from which

$$\mathbf{x} = (\mathbf{I} - \mathbf{B}')^{-1}\mathbf{v} \tag{12.9}$$

Since[2] $\mathbf{G}' = (\mathbf{I} - \mathbf{B}')^{-1}$, (12.9) is

$$\mathbf{x} = \mathbf{G}'\mathbf{v} \tag{12.10}$$

This is the version of the model that we will use in what follows. However, many analysts use the form in (12.6) and (12.7). Again, in terms of *changes* in $\mathbf{v}$ we would have

$$\Delta\mathbf{x} = \mathbf{G}'(\Delta\mathbf{v}) \tag{12.11}$$

The basic assumption of the supply-side approach is that the output distributions in $b_{ij}$ are stable in an economic system, meaning that if output of sector $i$ is, say, doubled, then the sales from $i$ to each of the sectors that purchase from $i$ will also be doubled. Instead of fixed input coefficients, fixed output coefficients are assumed in the supply-side model.

[2] This follows from matrix algebra results that $(\mathbf{A} \pm \mathbf{B})' = \mathbf{A}' \pm \mathbf{B}'$ and $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

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For sector $j$ in the $n$-sector case, from (12.10) we have

$$x_j = v_1 g_{1j} + \cdots + v_i g_{ij} + \cdots + v_n g_{nj} \tag{12.12}$$

Recall the typical equation in the solution to the demand-driven model, from (2.15) in Chapter 2:

$$x_i = l_{i1} f_1 + \cdots + l_{ij} f_j + \cdots + l_{in} f_n$$

The effect on output of sector $i$, $\Delta x_i$, of a \$1.00 change in final demand for sector $j$ goods ($\Delta f_j = 1$), is given by $l_{ij}$. (Again, for readers who are familiar with differential calculus, $\partial x_i/\partial f_j = l_{ij}$.) Column sums of $\mathbf{L} = [l_{ij}]$ were seen (Chapter 6) to be output multipliers; $\sum_{i=1}^{n} l_{ij}$ denotes the total new output throughout all $n$ sectors of the economy that is associated with a \$1.00 increase in final demand for sector $j$. Row sums of $\mathbf{L}$ can also be interpreted; $\sum_{j=1}^{n} l_{ij}$ shows the total new sector $i$ intermediate sales to all sectors that would be needed if there were a \$1.00 increase in the final demands for the outputs of *each* of the $n$ sectors in the economy.

From (12.12), the effect on sector $j$ output, $\Delta x_j$, of a \$1.00 change in the availability of primary inputs to sector $i$ ($\Delta v_i = 1$) is given by $g_{ij}$. (In calculus terms, $\partial x_j/\partial v_i = g_{ij}$; note that the order of the subscripts in this partial derivative is the opposite of that for $l_{ij}$). For example, if $g_{ij} = 0.67$, this has been interpreted to mean that if there is \$1.00 less labor available to sector $i$ as an input to production (due, say, to a strike), then the amount of reduction in sector $j$ output will be \$0.67. The reduction comes about because, in the input–output framework, a decrease in the available labor to sector $i$ means a decrease in sector $i$ output and hence in the outputs of all sectors that depend on sector $i$'s product as an input to their own production processes. This represents the same kind of effect, originating in an exogenous supply change, as is captured in the usual input–output system, which responds to exogenous demand changes.

In this (early) view of the Ghosh model, row and column sums in the output inverse, $\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = [g_{ij}]$ were given interpretations that parallel those in the Leontief quantity model. Row sums, $\sum_{j=1}^{n} g_{ij} = g_{i1} + \cdots + g_{in}$ $(=\partial x_1/\partial v_i + \cdots + \partial x_n/\partial v_i)$, were taken to represent the effect on total output throughout all sectors of the economy that would be associated with a \$1.00 change in primary inputs for sector $i$. This is the supply-side model's analog to an output (or demand) multiplier – a *column* sum in $\mathbf{L}$. These supply model *row* sums were termed input (or supply) multipliers. Also, column sums, $\sum_{i=1}^{n} g_{ij} = g_{1j} + \cdots + g_{nj}$ $(= \partial x_j/\partial v_1 + \cdots + \partial x_j/\partial v_n)$, were interpreted as the total effect on sector $j$ output if there were a \$1.00 change in the supply of primary factors for *each* of the $n$ sectors in the economy. These *column* sums were the supply-side model's parallel to the *row* sums of $\mathbf{L}$ in the demand model.

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*Numerical Illustration (Hypothetical Data)* Let

$$\mathbf{Z} = \begin{bmatrix} 225 & 600 & 110 \\ 250 & 125 & 425 \\ 325 & 700 & 150 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1200 \\ 2000 \\ 1500 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 265 \\ 1200 \\ 325 \end{bmatrix}$$

Then

$$\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z} = \begin{bmatrix} 1/1200 & 0 & 0 \\ 0 & 1/2000 & 0 \\ 0 & 0 & 1/1500 \end{bmatrix} \begin{bmatrix} 225 & 600 & 110 \\ 250 & 125 & 425 \\ 325 & 700 & 150 \end{bmatrix}$$
$$= \begin{bmatrix} .188 & .5 & .092 \\ .125 & .063 & .213 \\ .217 & .467 & .1 \end{bmatrix}$$

and

$$\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = \begin{bmatrix} 1.484 & .982 & .383 \\ .316 & 1.418 & .367 \\ .521 & .971 & 1.394 \end{bmatrix} \text{ and } \mathbf{G}' = \begin{bmatrix} 1.484 & .316 & .521 \\ .982 & 1.418 & .971 \\ .383 & .367 & 1.394 \end{bmatrix}$$

Thus, for example, if there were \$100 less labor available for sector 1 production and \$300 less for both sector 2 and sector 3 production, we would find, as in (12.11),

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1.484 & .316 & .521 \\ .982 & 1.418 & .971 \\ .383 & .367 & 1.394 \end{bmatrix} \begin{bmatrix} -100 \\ -300 \\ -300 \end{bmatrix} = \begin{bmatrix} -399.53 \\ -815.06 \\ -566.47 \end{bmatrix}$$

These figures, $\Delta x_1 = -400$, $\Delta x_2 = -815$ and $\Delta x_3 = -566$, would then be interpreted as the amounts by which the outputs of the three sectors would be reduced, given the decreases in labor inputs to the sectors.

If $\Delta v_1 = 1$ and $\Delta v_2 = \Delta v_3 = 0$,

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1.484 \\ .982 \\ .383 \end{bmatrix}$$

These figures represent the total additional outputs possible in each of the three sectors due to the availability of one more unit of primary inputs to sector 1. If $\Delta v_1 = -1$ and $\Delta v_2 = \Delta v_3 = 0$, these numbers will be negative, representing reduced output in the sectors. As suggested above, the sum of the elements in row 1 of $\mathbf{G}$ [column 1 of $\mathbf{G}'$], 2.849, represents the total potential impact throughout the economy of a \$1.00 change in the availability of primary inputs to sector 1. Again, as was noted above, this is parallel to the concept of the output multiplier for sector 1 in the ordinary, demand-driven input–output model. It is, in the context of this supply-side model, an *input* multiplier for sector 1. Similarly this kind of input multiplier for sector 2 is 2.101 and for sector 3 it is 2.886. In this view of the supply-side model, one might use these figures to decide where an additional dollar's worth of provision of primary resources (labor, etc.) would be most beneficial to the total economy, in terms of potential for supporting expanded output. Conversely, these input multipliers can indicate the potential contracting effects of shortages in primary inputs to a particular sector. From this point of view, a reduction by \$1.00 in the availability of a scarce resource could lead to a reduction in economy-wide output of \$2.849, \$2.101, or \$2.886, depending on where the primary input reduction occurs.

*Numerical Application (US Data)* Giarratani (1978) presents an application of the Ghosh model. He calculated output coefficients, **B**, and the associated output inverse matrix, **G**, using 78-sector 1967 US data. Supply multipliers ranged from a high of 4.01 for iron and ferroalloy-ores mining to 1.09 for medical and educational services and nonprofit organizations. With rankings of sectors such as this, it is possible to determine where primary factor constraints would have the greatest potential for limiting aggregate economic output – for example, a contemplated labor strike in one or more sectors.

Looking down the $j$th column of **G** allows one to identify supply linkages that have potential for significantly limiting the output of sector $j$. Among others, Giarratani considered an energy sector, petroleum refining and related industries (sector 31, the only secondary energy sector in the 78-sector 1967 US table). Examination of column 31 in the output inverse identifies the following among the largest coefficients: for sector 8, crude petroleum and natural gas, $g_{8,31} = 0.8605$; for sector 27, chemicals and chemical products, $g_{27,31} = 0.0513$; and for sector 12, maintenance and repair construction, $g_{12,31} = 0.0504$. The suggested interpretation is that interruptions in primary inputs to these sectors have the largest potential for disruptions in refined petroleum output.

Other examples of this kind of empirical analysis using the Ghosh model include Chen and Rose (1986) on the role of bauxite as a critical input in the Taiwanese economy and Davis and Salkin (1984) on the importance of water as an input in a county in California.

### *12.1.2 Relationships between* A *and* B *and between* L *and* G

Given $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ and $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$, $\mathbf{Z} = (\hat{\mathbf{x}})\mathbf{B}$; putting this into the definition of **A**,

$$\mathbf{A} = \hat{\mathbf{x}}\mathbf{B}\hat{\mathbf{x}}^{-1} \tag{12.13}$$

(When two matrices, **P** and **Q**, are connected by the relation $\mathbf{P} = \mathbf{M}\mathbf{Q}\mathbf{M}^{-1}$, they are said to be *similar*; this is denoted $\mathbf{P} \sim \mathbf{Q}$. Thus we see that **A** and **B** are similar matrices.) Of course, it also follows straightforwardly that

$$\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}} \tag{12.14}$$

Recall in section 6.6.2 on elasticities that element $(i, j)$ in the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$ was shown to capture the *direct* effect on industry $i$'s output (percentage change) resulting from a one percent change in industry $j$'s output. This was termed a *direct output-to-output* elasticity. Hence, from (12.14), these elasticities are precisely the elements in $\mathbf{B} = [b_{ij}]$.[3]

Consider $(\mathbf{I} - \mathbf{A})$. From (12.13), $(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \hat{\mathbf{x}}\mathbf{B}\hat{\mathbf{x}}^{-1}$. Since $\hat{\mathbf{x}}\mathbf{I}\hat{\mathbf{x}}^{-1} = \mathbf{I}$,

$$(\mathbf{I} - \mathbf{A}) = \hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})\hat{\mathbf{x}}^{-1}$$

That is, $(\mathbf{I} - \mathbf{A}) \sim (\mathbf{I} - \mathbf{B})$. Using a basic result on the inverse of a product of matrices – $(\mathbf{PQR})^{-1} = \mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{P}^{-1}$ – we find that since $(\mathbf{I} - \mathbf{A})^{-1} = [\hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})\hat{\mathbf{x}}^{-1}]^{-1}$

$$(\mathbf{I} - \mathbf{A})^{-1} = \hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})^{-1}\hat{\mathbf{x}}^{-1} \tag{12.15}$$

or

$$\mathbf{L} = \hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1} \tag{12.16}$$

Thus $\mathbf{L} \sim \mathbf{G}$. [The interested reader might confirm these similarity relationships in (12.13) and (12.15) for the small numerical illustration above.] The results in (12.16) can equally well be written as

$$\mathbf{G} = \hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}} \tag{12.17}$$

Again referring to section 6.6.2, we saw that element $(i, j)$ in the matrix $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ gives the percent increase in industry $i$ total output due to an initial exogenous one percent increase in industry $j$ output – the *total output-to-output elasticity* of industry $i$ output with respect to output in industry $j$. From (12.17) these elasticities are exactly the elements in $\mathbf{G} = [g_{ij}]$.

From these results it is clear that any measures defined for $\mathbf{A}$ – such as output multipliers or backward linkages (section 12.2.1) – can be found from $\mathbf{B}$, provided that $\mathbf{x}$ is also known. Conversely, input multipliers or forward linkages (section 12.2.2) – defined on $\mathbf{B}$ – can be found using $\mathbf{A}$ and $\mathbf{x}$.[4]

### *12.1.3 Comments on the Early Interpretation*

An early application of the Ghosh model is to be found in Augustinovics (1970), where direct-input coefficients ($\mathbf{A}$) and direct-output coefficients ($\mathbf{B}$) are compared for a number of countries and over time. However, reservations to this model began to appear in the early 1980s – for example in Giarratani (1980, 1981). The issue is: essentially what kind of economic behavior is represented by a system with constant supply distribution patterns? Ghosh had in mind the context of a planned economy experiencing severe excess demand, with government-imposed restrictions on supply patterns. This is probably not a very general situation in much of the modern world. However, Giarratani (1981, p. 283) suggested a possibly broader context:

[3] Using this interpretation, de Mesnard (2001) refers to the $a_{ij}$ and $b_{ij}$ coefficients as reflecting the *absolute* and *relative* direct influence of sector $j$ on sector $i$, respectively.

[4] It is easily shown that $\mathbf{A}$ and $\mathbf{B}$ have the same main diagonal elements; the same is true for $\mathbf{L}$ and $\mathbf{G}$. Using $\hat{\mathbf{M}}$ to denote the diagonal matrix whose elements are the main diagonal of a square matrix $\mathbf{M}$, $\hat{\mathbf{A}} = \hat{\mathbf{x}}\hat{\mathbf{B}}\hat{\mathbf{x}}^{-1} = \hat{\mathbf{B}}$, from (12.13), since order of multiplication of diagonal matrices makes no difference and $\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} = \mathbf{I}$. Exactly the same line of argument shows that $\hat{\mathbf{L}} = \hat{\mathbf{G}}$.

More interesting perhaps is the prospect that this behavior may be the result of voluntary supply decisions in the same context or, alternatively, given the disruption of some basic commodity. Firms may well attempt to maintain their existing markets … by allocating available product on the basis of deliveries in more normal times. Casual evidence on the U.S. experience would seem to support this hypothesis.

It was in this spirit that the application noted above (Giarratani, 1978) was carried out.

Oosterhaven (1980) raised reservations about the plausibility of the Ghosh model and then in the late 1980s a more vigorous exchange took place, particularly in Oosterhaven (1988, 1989), Gruver (1989) and Rose and Allison (1989). In essence, the problem is that primary input increases in sector $j$ are transmitted forward in the Ghosh model to output increases in all sectors that buy from $j$, without any corresponding increases in primary input use in those sectors. This is because $\Delta\mathbf{v}$ is viewed as exogenous and (in this example) is fixed at $\Delta\mathbf{v}' = [0, \ldots, 0, \Delta v_j, 0, \ldots, 0]$. This wreaks havoc with the notion of sectoral production functions where material inputs *plus primary inputs* are used in fixed proportions.

### 12.1.4 Joint Stability

*The Issue* When the demand-driven input–output model is used in standard fashion for impact analysis – as in $\Delta\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\Delta\mathbf{f}$ – a crucial assumption is that the direct-input coefficients matrix, $\mathbf{A}$, remains constant. As a consequence of the connections between $\mathbf{A}$ and $\mathbf{B}$, or between $\mathbf{L}$ and $\mathbf{G}$, this means that *in general* $\mathbf{B}$ (and therefore $\mathbf{G}$) *cannot* remain constant. This came to be known as the "joint stability" problem.[5] A numerical example illustrates the problem nicely. From the data for the three-sector hypothetical illustration in section 12.1.1, we also find[6]

$$\mathbf{A} = \begin{bmatrix} .188 & .3 & .073 \\ .208 & .063 & .283 \\ .271 & .35 & .1 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} 1.484 & .589 & .306 \\ .527 & 1.418 & .489 \\ .651 & .729 & 1.394 \end{bmatrix}$$

It will be useful at this point to use superscripts "0" to represent the base-year data, i.e., the given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{L}$, and $\mathbf{G}$ matrices, as well as the initial output, $\mathbf{x}$, will be denoted $\mathbf{A}^0$, $\mathbf{B}^0$ and so forth. Vectors and matrices that result from some exogenous change will be given superscripts "1". Suppose, for illustration, $\Delta\mathbf{f} = \begin{bmatrix} 100 \\ 40 \\ 30 \end{bmatrix}$; using the demand-driven model – $\Delta\mathbf{x} = \mathbf{L}^0\Delta\mathbf{f}$ – we find $\Delta\mathbf{x}(d) = \begin{bmatrix} 181.166 \\ 124.057 \\ 136.095 \end{bmatrix}$ and $\mathbf{x}^1(d) = \begin{bmatrix} 1381.2 \\ 2124.1 \\ 1636.1 \end{bmatrix}$. [We use $(d)$ to indicate that these are results from the demand-driven model, in which a constant $\mathbf{A}$ matrix is assumed.] From these results, we find the new transactions matrix

[5] See, among others, Dietzenbacher (1989), Miller (1989), Rose and Allison (1989), and Chen and Rose (1991).
[6] These matrices, along with $\mathbf{B}$ and $\mathbf{G}$ in section 12.1.1, illustrate the relationships shown in footnote 2, above.

associated with **A** and the new outputs; namely,

$$\mathbf{Z}^1(d) = \mathbf{A}^0[\hat{\mathbf{x}}^1(d)] = \begin{bmatrix} 258.969 & 637.217 & 119.980 \\ 287.743 & 132.754 & 463.560 \\ 374.066 & 743.420 & 163.610 \end{bmatrix}$$

[The reader can easily check that $\mathbf{Z}^1(d)\mathbf{i} + \mathbf{f}^1 = \mathbf{x}^1(d)$.] The direct-output coefficients matrix associated with these new transactions and new total outputs is found, as in (12.1), as

$$\mathbf{B}^1 = [\hat{\mathbf{x}}^1(d)]^{-1}\mathbf{Z}^1(d) = \begin{bmatrix} .188 & .461 & .087 \\ .136 & .063 & .218 \\ .229 & .454 & .1 \end{bmatrix}$$

Recall from above that

$$\mathbf{B}^0 = \begin{bmatrix} .188 & .5 & .092 \\ .125 & .063 & .213 \\ .217 & .467 & .1 \end{bmatrix}$$

and clearly $\mathbf{B}^1 \neq \mathbf{B}^0$. [One simple measure of the difference is the average of all of the (absolute) percentage differences – $(1/n^2)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left|b_{ij}^0 - b_{ij}^1\right|/b_{ij}^0\right] \times 100$. Here this is 3.58 percent.] The upshot is that, at least in this example (but actually in general), the assumption of a constant **A** matrix, used in an impact analysis, carries with it the requirement that **B** change as a result of the impact.

An exactly similar problem occurs if one uses the supply-driven model to assess the impact of a change in primary inputs. For example, from the data for this three-sector example, $\mathbf{v}' = \begin{bmatrix} 400 & 575 & 815 \end{bmatrix}$. Suppose that $(\Delta\mathbf{v})' = \begin{bmatrix} 50 & 100 & 20 \end{bmatrix}$; using (12.11) to assess the output effects of this change in primary inputs, we find $[\Delta\mathbf{x}(s)]' = \begin{bmatrix} 116.221 & 210.325 & 83.720 \end{bmatrix}$ and $[\mathbf{x}^1(s)]' = \begin{bmatrix} 1316.2 & 2210.3 & 1583.7 \end{bmatrix}$ [Now $(s)$ denotes results from the supply-driven model.] Parallel to the demand-driven example, there is now a new transactions matrix,

$$\mathbf{Z}^1(s) = [\hat{\mathbf{x}}^1(s)]\mathbf{B}^0 = \begin{bmatrix} 246.792 & 658.111 & 120.654 \\ 276.291 & 138.145 & 469.694 \\ 343.139 & 739.069 & 158.372 \end{bmatrix}$$

In conjunction with the associated $\mathbf{x}^1(s)$, this $\mathbf{Z}^1(s)$ defines the corresponding direct-input coefficients matrix, $\mathbf{A}^1$, namely

$$\mathbf{A}^1 = \mathbf{Z}^1(s)[\hat{\mathbf{x}}^1(s)]^{-1} = \begin{bmatrix} .188 & .3 & .076 \\ .210 & .063 & .3 \\ .261 & .334 & .1 \end{bmatrix}$$

Originally,

$$\mathbf{A}^0 = \begin{bmatrix} .188 & .3 & .073 \\ .208 & .063 & .283 \\ .271 & .35 & .1 \end{bmatrix}$$

and $\mathbf{A}^1 \neq \mathbf{A}^0$. (In this case, the average absolute difference is 2.06 percent.)[7]

This apparent inconsistency – the fact that the requirement of a constant $\mathbf{A}$ (for demand-driven model impact analysis) implies a non-constant $\mathbf{B}$ in the related supply-driven model or that the constant $\mathbf{B}$ needed for supply-driven model impact analysis carries with it the implication of a non-constant $\mathbf{A}$ in the related demand-driven model – led to several empirical studies on relative joint stability (see, for example, Rose and Allison, 1989, or Chen and Rose, 1991). In general, the conclusion drawn was that instability in actual empirical applications was not a major issue.

*Conditions under which both* A *and* B *will be Stable* Assume that we have found the new outputs resulting from new final demands using the demand-driven model – $\mathbf{x}^1 = (\mathbf{I} - \mathbf{A}^0)\mathbf{f}^1$, so $\mathbf{A}^1 = \mathbf{A}^0$. From (12.14),

$$\mathbf{B}^1 = (\hat{\mathbf{x}}^1)^{-1}\mathbf{A}^0\hat{\mathbf{x}}^1$$

and substituting $\mathbf{A}^0$ from (12.13)

$$\mathbf{B}^1 = (\hat{\mathbf{x}}^1)^{-1}\hat{\mathbf{x}}^0\mathbf{B}^0(\hat{\mathbf{x}}^0)^{-1}\hat{\mathbf{x}}^1$$

Let $\hat{\mathbf{e}} = \hat{\mathbf{x}}^1(\hat{\mathbf{x}}^0)^{-1}$ where $e_i = x_i^1/x_i^0$ can be thought of as a kind of "growth rate" for sector $i$ (remember that order of multiplication makes no difference when the matrices are diagonal); then

$$\mathbf{B}^1 = \hat{\mathbf{e}}^{-1}\mathbf{B}^0\hat{\mathbf{e}}$$

A similar story holds if the supply-driven model is used, with $\mathbf{B}^1 = \mathbf{B}^0$; namely

$$\mathbf{A}^1 = \hat{\mathbf{e}}\mathbf{A}^0\hat{\mathbf{e}}^{-1}$$

If each sector's output changes at the same rate – $e_i = x_i^1/x_i^0 = \lambda$ for all $i$ – then $\hat{\mathbf{e}} = \lambda\mathbf{I}$ and $\mathbf{B}^1 = [(1/\lambda)\mathbf{I}]\mathbf{B}^0(\lambda\mathbf{I}) = \mathbf{B}^0$. A similar argument shows that $\mathbf{A}^1 = \mathbf{A}^0$ under the same conditions, after an impact analysis with the supply-driven model.[8]

### *12.1.5 Reinterpretation as a Price Model*

In order to overcome the criticisms and implausibilities in the original view of the Ghosh model, Dietzenbacher (1997) proposed an alternative interpretation by suggesting that the model be viewed not as a *quantity* model but as a *price* model (see also extensive discussions on alternative interpretations of the Ghosh model in Oosterhaven, 1996 and de Mesnard, 2007). We illustrate the idea by looking again at results from the numerical example in the previous section. Specifically, for

$$(\mathbf{v}^1)' = (\mathbf{v}^0)' + (\Delta\mathbf{v})' = \begin{bmatrix}400 & 575 & 815\end{bmatrix} + \begin{bmatrix}50 & 100 & 20\end{bmatrix} = \begin{bmatrix}450 & 675 & 835\end{bmatrix}$$

[7] For exactly the same reasons as shown in footnote 3, above, $\hat{\mathbf{A}}^0 = \hat{\mathbf{A}}^1$ and $\hat{\mathbf{B}}^0 = \hat{\mathbf{B}}^1$. These relationships are illustrated by the matrices in this section.

[8] For much more detail on these matters see Dietzenbacher (1989, 1997).

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 551.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=585

we found, using $\mathbf{x}^1(s) = (\mathbf{G}^0)'\mathbf{v}^1$ [(12.10)],

$$[\mathbf{x}^1(s)]' = \begin{bmatrix} 1316.2 & 2210.3 & 1583.7 \end{bmatrix}$$

Suppose that we view the elements in the supply-driven model not as *quantities* (in which case elements in $\Delta\mathbf{v}$ are interpreted as changes in the *amounts* of primary inputs available to the economy and elements in $\Delta\mathbf{x}$ are interpreted in changes in *quantities* produced) but rather as *values* (in which case elements in $\Delta\mathbf{v}$ reflect changes in the *prices* or *costs* of primary inputs and elements in $\Delta\mathbf{x}$ reflect changes in the *values* of outputs). In the demand-driven model of earlier chapters all prices are assumed fixed in an impact analysis and quantities change as a result of changes in the quantities of final demands. Now we assume that all quantities are fixed and use the Ghosh model to assess the repercussions throughout the economy of changes in primary input prices. In that reinterpretation, we can use the term *Ghosh price model*, which can reasonably be looked upon as a *cost-push input–output model*. Changes in primary input costs are transmitted throughout the economy as they are passed on (completely) by producers in the prices of their products that are purchased by other intermediate users, who in turn increase their prices accordingly, etc.

With this interpretation, we identify the *relative* price changes easily as the ratios of elements in $\mathbf{x}^0$ to those in $\mathbf{x}^1(s)$, since *quantities* are fixed and only *valuations* change. Define $\boldsymbol{\pi}$ as the vector of these price ratios,

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}[\mathbf{x}^1(s)] \tag{12.18}$$

where $\pi_j = x_j^1(s)/x_j^0 = p_j^1 q_j^0/p_j^0 q_j^0 = p_j^1/p_j^0$ (where $q_j^0$ is a physical measure of the output of sector $j$ in the base period). For this three-sector example,

$$\boldsymbol{\pi} = \begin{bmatrix} x_1^1(s)/x_1^0 \\ x_2^1(s)/x_2^0 \\ x_3^1(s)/x_3^0 \end{bmatrix} = \begin{bmatrix} 1316.2/1200 \\ 2210.3/2000 \\ 1583.7/1500 \end{bmatrix} = \begin{bmatrix} 1.0968 \\ 1.1052 \\ 1.0558 \end{bmatrix} \tag{12.19}$$

This indicates that (unit) prices of the products of sectors 1, 2, and 3 would rise by 9.68, 10.52, and 5.58 percent, respectively, in response to primary input cost increases of 12.5 [=(50/400) × 100] percent, 17.39 [= (100/575) × 100] percent and 2.45 [=(20/815) × 100] percent, for the three sectors respectively.

Similarly, when $\Delta v_1 = 1$ and $\Delta v_2 = \Delta v_3 = 0$, we found $\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1.484 \\ .982 \\ .383 \end{bmatrix}$ so $\mathbf{x}^1(s) = \begin{bmatrix} 1201.48 \\ 2000.98 \\ 1500.38 \end{bmatrix}$. This can now be interpreted in terms of price ratios for the three sectors of

$$\boldsymbol{\pi} = \begin{bmatrix} 1201.48/1200 \\ 2000.98/2000 \\ 1500.38/1500 \end{bmatrix} = \begin{bmatrix} 1.0012 \\ 1.0005 \\ 1.0003 \end{bmatrix}$$

This says that prices would be expected to increase by 0.12, 0.05, and 0.03 percent in the three sectors, respectively, in the face of a 0.25 percent [=(401/400)×100] increase in the cost of primary inputs to sector 1 only.

*Connection to the Leontief Price Model (Algebra)* It is straightforward to show that the Ghosh price model and the Leontief price model (section 2.6) generate exactly the same results. The Ghosh price model finds

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}[\mathbf{x}^1(s)]$$

Since $[\mathbf{x}^1(s)] = (\mathbf{G}^0)'(\mathbf{v}^1)$[(12.10)],

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}(\mathbf{G}^0)'\mathbf{v}^1$$

From $\mathbf{G} = \hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ in (12.17), $\mathbf{G}' = \hat{\mathbf{x}}\mathbf{L}'\hat{\mathbf{x}}^{-1}$, and so we have

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}[\hat{\mathbf{x}}^0(\mathbf{L}^0)'(\hat{\mathbf{x}}^0)^{-1}]\mathbf{v}^1 = (\mathbf{L}^0)'(\hat{\mathbf{x}}^0)^{-1}\mathbf{v}^1$$

Finally, since primary input *coefficients* are found as $v_{ci}^1 = v_i^1/x_i^0$, or $\mathbf{v}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{v}_1$,

$$\boldsymbol{\pi} = (\mathbf{L}^0)'(\hat{\mathbf{x}}^0)^{-1}\hat{\mathbf{x}}^0\mathbf{v}_c^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 \tag{12.20}$$

In the Leontief price model of section 2.6, it is also the case that primary input price changes generate relative price changes [as in (2.33), which is repeated below]:

$$\tilde{\mathbf{p}} = [\mathbf{I} - (\mathbf{A}^0)']^{-1}\mathbf{v}_c^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 \tag{12.21}$$

As (12.20) and (12.21) make clear, $\boldsymbol{\pi} = \tilde{\mathbf{p}}$. The Leontief price (cost-push) model (section 2.6) and the Ghosh price (cost-push) model generate the same results; the former directly in terms of the vector of relative price changes, $\tilde{\mathbf{p}}$, and the latter in terms of new outputs, $\mathbf{x}^1(s)$, from which $\boldsymbol{\pi}$ is found as the ratio of new to old output values.

*Connection to the Leontief Price Model (Numerical Illustration)* Using data from the hypothetical example in section 12.1.1 and 12.1.5, we find the base-year primary input coefficients as

$$\mathbf{v}_c^0 = \begin{bmatrix} 400/1200 \\ 575/2000 \\ 815/1500 \end{bmatrix} = \begin{bmatrix} .3333 \\ .2875 \\ .5433 \end{bmatrix}$$

As expected,

$$\tilde{\mathbf{p}}^0 = (\mathbf{L}^0)'\mathbf{v}_c^0 = \begin{bmatrix} 1.4840 & 0.5266 & 0.6514 \\ 0.5893 & 1.4179 & 0.7287 \\ 0.3064 & 0.4893 & 1.3936 \end{bmatrix} \begin{bmatrix} .3333 \\ .2875 \\ .5433 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

verifies that all prices are one ("per dollar's worth of output") in the base-year Leontief model.

Now consider the primary input price increases from the example above, namely

$$(\mathbf{v}^1)' = (\mathbf{v}^0)' + (\Delta\mathbf{v})' = \begin{bmatrix}400 & 575 & 815\end{bmatrix} + \begin{bmatrix}50 & 100 & 20\end{bmatrix} = \begin{bmatrix}450 & 675 & 835\end{bmatrix}$$

In terms of primary input *coefficients*, we have

$$\mathbf{v}_c^1 = \begin{bmatrix}450/1200 \\ 675/2000 \\ 835/1500\end{bmatrix} = \begin{bmatrix}.3750 \\ .3375 \\ .5566\end{bmatrix}$$

and using (12.21),

$$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 = \begin{bmatrix}1.4840 & 0.5266 & 0.6514 \\ 0.5893 & 1.4179 & 0.7287 \\ 0.3064 & 0.4893 & 1.3936\end{bmatrix}\begin{bmatrix}.3750 \\ .3375 \\ .5566\end{bmatrix} = \begin{bmatrix}1.0968 \\ 1.1051 \\ 1.0558\end{bmatrix}$$

As expected, these are precisely the same results as we found above for $\boldsymbol{\pi}$ in (12.19).

Either exercise produces the result that price increases of 9.68, 10.51, and 5.58 percent are to be expected for the output of the three sectors as a result of the primary input cost increases given in $(\Delta\mathbf{v})' = \begin{bmatrix}50 & 100 & 20\end{bmatrix}$.

*A Ghosh Quantity Model* Thus far we have seen a Leontief quantity model, a Leontief price model, and a Ghosh price model. It is logical to expect that a Ghosh quantity model also exists (Dietzenbacher, 1997). From the familiar Leontief quantity model, $\mathbf{x}^1 = \mathbf{L}^0\mathbf{f}^1$, and $\mathbf{L}^0 = \hat{\mathbf{x}}^0\mathbf{G}^0(\hat{\mathbf{x}}^0)^{-1}$ [(12.16)], we have

$$\mathbf{x}^1 = \hat{\mathbf{x}}^0\mathbf{G}^0(\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1$$

Define the new final-demands as proportions (coefficients) of base-period outputs – $(f_c^1)_i = [f_i^1/x_i^0]$ and $\mathbf{f}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1$ – and premultiply both sides by $(\hat{\mathbf{x}}^0)^{-1}$,

$$\tilde{\mathbf{x}} = (\hat{\mathbf{x}}^0)^{-1}\mathbf{x}^1 = (\hat{\mathbf{x}}^0)^{-1}\hat{\mathbf{x}}^0\mathbf{G}^0(\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1 = \mathbf{G}^0\mathbf{f}_c^1$$

where $\tilde{x}_i = x_i^1/x_i^0$. In this case changes in final-demand *proportions* (of gross outputs) are translated into relative output measures; that is, an *index* showing new outputs, $\mathbf{x}^1$, as proportions of base-period outputs, $\mathbf{x}^0$.

This is the straightforward algebraic derivation of a Ghosh quantity model. The reader can explore the logic of the "story" behind it. Table 12.1 gathers together some of the relevant information about these four models. The quantity and price models – either Leontief or Ghosh – are often described as "dual" to each other[9], while the Leontief variant of the quantity model has been described as the "mirror image" of the Ghosh quantity model, and similarly for the Leontief and Ghosh price models. (Some of this material appeared earlier in Table 2.13.)

[9] There are some rather detailed mathematical discussions on what constitutes a pair of "dual" models. For our input–output models we simply take the term to mean that one model determines quantities (with prices fixed), the other determines prices (with quantities fixed) and the fundamental structural relationships (in $\mathbf{L}^0$ or in $\mathbf{G}^0$) are at the heart (although transposed) of each model and its dual.

**Table 12.1** Overview of the Leontief and Ghosh Quantity and Price Models

| Model | | Leontief | Ghosh |
|---|---|---|---|
| Price (Cost-push) [Quantities fixed; prices change] | Exogenous Variables | $\mathbf{v}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{v}^1 = [v_j^1/x_j^0]$ | $\mathbf{v}^1 = [v_j^1]$ |
| | Endogenous Variables | $\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1$ $[\tilde{p}_i = x_i^1(d)/x_i^0]$ | $\mathbf{x}^1(s) = (\mathbf{G}^0)'\mathbf{v}^1$ |
| | Coefficient Stability | $\mathbf{A}^1 \neq \mathbf{A}^0$ | $\mathbf{B}^1 = \mathbf{B}^0$ |
| Quantity (Demand-pull) [Prices fixed; quantities change] | Exogenous Variables | $\mathbf{f}^1 = [f_i^1]$ | $\mathbf{f}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1 = [f_i^1/x_i^0]$ |
| | Endogenous Variables | $\mathbf{x}^1(d) = \mathbf{L}^0\mathbf{f}^1$ | $\tilde{\mathbf{x}} = \mathbf{G}^0\mathbf{f}_c^1$ $[\tilde{x}_i = x_i^1(s)/x_i^0]$ |
| | Coefficient Stability | $\mathbf{A}^1 = \mathbf{A}^0$ | $\mathbf{B}^1 \neq \mathbf{B}^0$ |

## 12.2 Linkages in Input–Output Models

In the framework of an input–output model, production by a particular sector has two kinds of economic effects on other sectors in the economy. If sector $j$ increases its output, this means there will be increased *demands* from sector $j$ (as a purchaser) on the sectors whose goods are used as inputs to production in $j$. This is the direction of causation in the usual demand-side model, and the term *backward linkage* is used to indicate this kind of interconnection of a particular sector with those ("upstream") sectors from which it purchases inputs. On the other hand, increased output in sector $j$ also means that additional amounts of product $j$ are available to be used as inputs to other sectors for their own production – that is, there will be increased *supplies* from sector $j$ (as a seller) for the sectors that use good $j$ in their production. This is the direction of causation in the supply-side model. The term *forward linkage* is used to indicate this kind of interconnection of a particular sector with those ("downstream") sectors to which it sells its output.

Measures have been proposed to quantify such backward and forward linkages, or economic "connectedness." Comparisons of the strengths of backward and forward linkages for the sectors in a single economy provide one mechanism for identifying "key" or "leading" sectors in that economy (those sectors that are most connected and therefore, in some sense, most "important") and for grouping sectors into spatial clusters. And if data are available for more than one time period, the evolution of these interconnections can be studied. Also, examination of these measures for similar sectors in different countries provides one method of making international comparisons of the structure of production.

If the backward linkage of sector $i$ is larger than that of sector $j$, one might conclude that a dollar's worth of expansion of sector $i$ output would be more beneficial to the economy than would an equal expansion in sector $j$'s output, in terms of the productive activity throughout the economy that would be generated by it. Similarly, if the forward linkage of sector $r$ is larger than that of sector $s$, it could be said that a dollar's worth of expansion of the output of sector $r$ is more essential to the economy than a similar expansion in the output of sector $s$, from the point of view of the overall productive activity that it could support.

There have been numerous suggestions for differing definitions and refinements of these linkage and key sector measures and others of economic connectedness. Early work includes Rasmussen (1957),[10] Hirschman (1958), Chenery and Watanabe (1958), Yotopoulos and Nugent (1973), Laumas (1975) and Jones (1976), and there has been, and continues to be, a good deal of discussion [for example, on the "proper" definition, see the debate among several authors in the May 1976 issue of the *Quarterly Journal of Economics*, or the Diamond (1976), Schultz and Schumacher (1976) and Laumas (1976a) exchange in *Kyklos*]. Questions on the exact role of linkage measures and the identification of key sectors in development planning have been raised in McGilvray (1977) and Hewings (1982), among others. Our purpose here is simply to introduce the reader to some of the most prevalent of these measures and, in particular, to indicate how they are derived from information in either the demand-side or the supply-side input–output model.

There also have been numerous suggestions for various ways of combining forward and backward linkage measures (examples can be found in Hübler, 1979; Loviscek, 1982; Meller and Marfán, 1981; Cella, 1984; Clements, 1990 and Adamou and Gowdy, 1990). Generally these combined measures have been superseded by the rankings that emerge from the "hypothetical extraction" approach, to which we turn in section 12.2.5.

### *12.2.1 Backward Linkage*

In its simplest form, a measure of the strength of the backward linkage of sector $j$ – the amount by which sector $j$ production depends on interindustry inputs – is given by the sum of the elements in the $j$th column of the direct input coefficients matrix, namely $\sum_{i=1}^{n} a_{ij}$. Since the coefficients in $\mathbf{A}$ are measures of direct effects only, this is called the *direct* backward linkage:[11]

$$BL(d)_j = \sum_{i=1}^{n} a_{ij} \tag{12.22}$$

[10] Hirschman (1958) cites an edition of this book (same title) published by Einar Harcks in Copenhagen in 1956. This must be a precursor to the 1957 North-Holland edition (also under the Einar Harcks imprint) which is identified as a "second printing."

[11] It would be more consistent with standard vector-matrix notation to use some lower-case designation such as $b_j$ for sector $j$'s backward linkage (a scalar), but $BL_j$ seems to have become standard.

In terms of transactions (**Z**, not **A**), this is simply the value of total intermediate inputs for sector $j$ $\left(\sum_{i=1}^{n} z_{ij}\right)$ as a proportion of the value of $j$'s total output ($x_j$). This definition, in transactions terms, was first proposed by Chenery and Watanabe (1958). If we define $\mathbf{b}(d) = [BL(d)_1, \ldots, BL(d)_n]$, then

$$\mathbf{b}(d) = \mathbf{i}'\mathbf{A} \tag{12.23}$$

To capture both direct and indirect linkages in an economy, column sums of the total requirements matrix, $\mathbf{L} = [l_{ij}]$, were proposed as a *total backward* linkage measure (Rasmussen, 1957); these are output multipliers (Chapter 6). For sector $j$ we have

$$BL(t)_j = \sum_{i=1}^{n} l_{ij} \tag{12.24}$$

The corresponding row vector of direct and indirect backward-linkage measures for each sector is

$$\mathbf{b}(t) = \mathbf{i}'\mathbf{L} \tag{12.25}$$

There is some disagreement in the literature on whether the on-diagonal elements in **A** or **L** should be included or netted out of the summations (see, for example, Harrigan and McGilvray, 1988). To the extent that these "internal linkages" constitute part of Hirschman's (1958, p. 100) "… input-provision, derived demand … effects," they are appropriately included. On the other hand, if one is specifically interested in a sector's "backward dependence" on or linkage to the *rest* of the economy, they should be omitted.

Also, various normalizations of these measures have been proposed and used in empirical studies. For example, let

$$\overline{BL}(d)_j = \frac{BL(d)_j}{(1/n)\sum_{j=1}^{n} BL(d)_j} = \frac{\sum_{i=1}^{n} a_{ij}}{(1/n)\sum_{i=1}^{n} a_{ij} \sum_{j=1}^{n} a_{ij}}$$

(where the overbar suggests a normalized measure). In this case, sector $j$'s backward linkage is divided by the (simple) average of all backward linkages. (Various weighted averages have also been suggested.) In (row) vector form, these normalized direct backward linkages are (note that $\mathbf{i}'\mathbf{A}\mathbf{i}$ is a scalar)

$$\bar{\mathbf{b}}(d) = \frac{\mathbf{i}'\mathbf{A}}{(\mathbf{i}'\mathbf{A}\mathbf{i})/n} = \frac{n\mathbf{i}'\mathbf{A}}{\mathbf{i}'\mathbf{A}\mathbf{i}} \tag{12.26}$$

The average value of $\bar{\mathbf{b}}(d)$ is unity – $[\bar{\mathbf{b}}(d)]\mathbf{i}(1/n) = \left[n\mathbf{i}'\mathbf{A}/\mathbf{i}'\mathbf{A}\mathbf{i}\right][\mathbf{i}/n] = 1$ – so that sectors with "above average" (stronger) direct backward linkages have indices that are greater than one and that those with "below average" (weaker) direct backward linkages have indices that are less than one. The same logic generates

$$\bar{\mathbf{b}}(t) = \frac{n\mathbf{i}'\mathbf{L}}{\mathbf{i}'\mathbf{L}\mathbf{i}} \tag{12.27}$$

as a normalized total backward linkage index, also with an average value of unity. (This is the "Index of the Power of Dispersion" suggested by Rasmussen, 1957.)

### *12.2.2 Forward Linkage*

An early measure of *direct forward linkage* was also proposed, based on **A** and **L**, as the row sums **Ai**, along with an associated *total forward linkage* measure, the row sums **Li**.[12] Both of these have been viewed with skepticism, because they are generated by a peculiar stimulus – a simultaneous increase of one unit in the gross outputs of every sector in the case of **Ai** and an increase of one unit in the final demands of every sector in the case of **Li**.[13]

This dissatisfaction led to the suggestion that elements from the Ghosh model would be more appropriate as forward linkage measures (Beyers, 1976; Jones, 1976). The row sums **Bi** were suggested as better measures of *direct forward linkage*. In terms of transactions (**Z**, not **B**), this is simply the value of total intermediate sales by sector $i\left(\sum_{j=1}^{n} z_{ij}\right)$ as a proportion of the value of $i$'s total output ($x_i$). (This also was first proposed in Chenery and Watanabe, 1958). In addition, row sums of the Ghosh inverse, $\mathbf{G} = [g_{ij}]$, were suggested as a better measure of *total forward linkages*. As with backward linkage measures, inclusion or exclusion of on-diagonal elements is an issue, and normalizations are usual.

Thus, the parallels to (12.22) and (12.24) for direct forward linkages are

$$FL(d)_i = \sum_{j=1}^{n} b_{ij} \tag{12.28}$$

and

$$FL(t)_i = \sum_{j=1}^{n} g_{ij} \tag{12.29}$$

In addition, the same two normalized versions for forward linkages can be found. Matrix expressions for all these results are collected in Table 12.2.

### *12.2.3 "Net" Backward Linkage*

Another linkage measure was proposed by Dietzenbacher (2005) in his interpretation of the content of the Oosterhaven and Stelder net multiplier formulation (section 6.5.3).

[12] In normalized form, $n\mathbf{Li}/\mathbf{i}'\mathbf{Li}$, this is Rasmussen's (1957) "Index of Sensitivity of Dispersion."
[13] Among the first to make an issue of weightings in linkage measures was Laumas (1976b). Others before him (e.g., Hazari, 1970; Diamond, 1974), however, had used sets of weights other than unit vectors.

**Table 12.2** Linkage Measures

| | $BL$ | $FL$ | $\overline{BL}$ | $\overline{FL}$ |
|---|---|---|---|---|
| Direct | $\mathbf{i}'\mathbf{A}$ | $\mathbf{Bi}$ | $\dfrac{n\mathbf{i}'\mathbf{A}}{\mathbf{i}'\mathbf{Ai}}$ | $\dfrac{n\mathbf{Bi}}{\mathbf{i}'\mathbf{Bi}}$ |
| Total | $\mathbf{i}'\mathbf{L}$ | $\mathbf{Gi}$ | $\dfrac{n\mathbf{i}'\mathbf{L}}{\mathbf{i}'\mathbf{Li}}$ | $\dfrac{n\mathbf{Gi}}{\mathbf{i}'\mathbf{Gi}}$ |

Start with the observation that $\mathbf{L\hat{f}}$ is a matrix whose $i, j$th element represents output of $i$ generated by $f_j$. Row sums of $\mathbf{L\hat{f}}$ are given by $\mathbf{L\hat{f}i} = \mathbf{Lf} = \mathbf{x}$; the $i$th element of this column vector is simply $x_i$, the output of $i$ generated by all final demands – the standard interpretation of $\mathbf{x}$. Column sums of $\mathbf{L\hat{f}}$ are given by $\mathbf{i}'\mathbf{L\hat{f}}$; the $j$th element of this row vector is the output needed from *all* sectors to satisfy $f_j$. The Oosterhaven–Stelder net output multiplier was defined as $\mathbf{i}'\mathbf{L\hat{f}}_c = \mathbf{i}'\mathbf{L\hat{f}\hat{x}}^{-1}$ (a row vector). Replacement of $\mathbf{\hat{x}}$ by $\langle \mathbf{L\hat{f}i} \rangle$ leads to

$$\mathbf{i}'\mathbf{L\hat{f}}_c = \mathbf{i}'\mathbf{L\hat{f}\hat{x}}^{-1} = (\mathbf{i}'\mathbf{L\hat{f}})\langle \mathbf{L\hat{f}i} \rangle^{-1}$$

The $j$th element in this row can be seen to be a ratio, namely

$$(\mathbf{i}'\mathbf{L\hat{f}}_c)_j = \frac{j\text{th column sum of } \mathbf{L\hat{f}}}{j\text{th row sum of } \mathbf{L\hat{f}}}$$

In words: the output generated in all industries by $f_j$ divided by the output generated in $j$ by all final demands. This suggests a kind of "net" backward linkage or net key sector measure. In particular, if $(\mathbf{i}'\mathbf{L\hat{f}}_c)_j > 1$ then economy-wide output generated by final demand in $j$ is larger than the amount of $j$'s output that is generated by all the other industries' final demands. So industry $j$ can be said to be more important for the others than the others are for industry $j$, and $j$ would thus be identified as a key sector by this measure.

### *12.2.4 Classifying Backward and Forward Linkage Results*

Studies that attempt to identify key sectors from their backward and forward linkage measures usually calculate both (generally in normalized form) and then select those sectors with a high score on both measures.[14] In normalized form, this means sectors with both backward and forward linkages greater than one.

Often, sectors are distributed over a four-way classification as (1) generally independent of (not strongly connected to) other sectors (both linkage measures less than 1), (2) generally dependent on (connected to) other sectors (both linkage measures greater than 1), (3) dependent on interindustry supply (only backward linkage greater than 1)

[14] There have been suggestions for "combined" measures to capture "total" linkage. For example, Hübler (1979) proposed column sums from $[\mathbf{I} - (0.5)(\mathbf{A} + \mathbf{B}')]^{-1}$ for this purpose. More comprehensive measures of total linkage come from hypothetical extraction approaches (section 12.2.6).

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 559.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=593

**Table 12.3** Classification of Backward and Forward Linkage Results

| | | Direct or Total Forward Linkage | |
|---|---|---|---|
| | | Low (<1) | High (>1) |
| Direct or Total Backward Linkage | Low (<1) | (I) Generally independent | (II) Dependent on interindustry demand |
| | High (>1) | (IV) Dependent on interindustry supply | (III) Generally dependent |

and (4) dependent on interindustry demand (only forward linkage greater than 1). This can be displayed in a $2 \times 2$ table, such as shown in Table 12.3.[15] With data for two or more time periods, a table of this sort for each period will give one indication of the evolution of the economy.[16]

### 12.2.5 Spatial Linkages

Exactly the same kinds of measures can be applied to multiregional input–output data to assess the types and intensities of spatial interdependence or connectedness. These address the issue of strength of economic connections among regions in an economy and, if data for more than one period are available, how those connections are changing over time – for example, increasing regional self-sufficiency or increasing interregional dependence. These measures can be aggregate – that is, is region $r$ in general dependent on imports or exports (or both) or relatively self-sufficient? Or they can be sector/region specific – assessing the import- or export-dependence of sector $i$ in region $r$ on one sector (or all sectors) in another region (or regions). Recalling that total backward linkage is measured by the output multiplier, it is clear that the interregional multipliers discussed in section 6.3 get at exactly these kinds of questions. (Early presentations of the spatial form of linkage measures are in Miller and Blair, 1988 and Batten and Martellato, 1988.)

In the two-region (nation) context, we have $\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix}$, $\mathbf{L} = \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{L}^{rs} \\ \mathbf{L}^{sr} & \mathbf{L}^{ss} \end{bmatrix}$ and $\mathbf{G} = \begin{bmatrix} \mathbf{G}^{rr} & \mathbf{G}^{rs} \\ \mathbf{G}^{sr} & \mathbf{G}^{ss} \end{bmatrix}$. One straightforward set of spatial linkage measures closely parallels the sectoral linkage cases above. In this two-region case, the direct backward linkage of sector $j$ in region $r$ will have both an intraregional and an interregional

[15] This two-way table arrangement appears to have originated with Chenery and Watanabe (1958).

[16] Further subtleties are possible. Each quadrant can be further subdivided: for example, quadrant III could be divided into four more categories, those above and those below one standard deviation above the mean (of 1). And similarly for the other three numbered quadrants.

component. Specifically,

$$BL(d)_j^r = BL(d)_j^{rr} + BL(d)_j^{sr} = \sum_{i=1}^{n} a_{ij}^{rr} + \sum_{i=1}^{n} a_{ij}^{sr}$$

One measure of the relative strength of intra- vs. interregional (internal vs. external) direct backward dependence is given by the percentages

$$100[BL(d)_j^{rr}/BL(d)_j^r] \quad \text{and} \quad 100[BL(d)_j^{sr}/BL(d)_j^r]$$

or, using an alternative normalization,

$$BL(d)_j^{rr}/x_j^r \quad \text{and} \quad BL(d)_j^{sr}/x_j^r$$

Parallels can be found for total backward linkages, namely

$$BL(t)_j^r = BL(t)_j^{rr} + BL(t)_j^{sr} = \sum_{i=1}^{n} l_{ij}^{rr} + \sum_{i=1}^{n} l_{ij}^{sr}$$

and

$$100[BL(t)_j^{rr}/BL(t)_j^r] \quad \text{and} \quad 100[BL(t)_j^{sr}/BL(t)_j^r]$$
$$BL(t)_j^{rr}/x_j^r \text{ and } BL(t)_j^{sr}/x_j^r$$

In compact matrix form, direct and total intra- and interregional backward linkages for each sector in region $r$ are given by the $n$ elements in the following vectors [the parallels are (12.23) and (12.25)]

$$\mathbf{b}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr} \quad \text{and} \quad \mathbf{b}(d)^{sr} = \mathbf{i}'\mathbf{A}^{sr}$$
$$\mathbf{b}(t)^{rr} = \mathbf{i}'\mathbf{L}^{rr} \quad \text{and} \quad \mathbf{b}(t)^{sr} = \mathbf{i}'\mathbf{L}^{sr}$$

and

$$\mathbf{b}(d)^r = \mathbf{b}(d)^{rr} + \mathbf{b}(d)^{sr} \quad \text{and} \quad \mathbf{b}(t)^r = \mathbf{b}(t)^{rr} + \mathbf{b}(t)^{sr}$$

Ignoring the sectoral detail, one aggregate measure of a region's direct and total backward linkage to itself and to the other region(s) is found by summing (or averaging) over all sectors. For example,

$$B(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}\mathbf{i} \quad \text{or} \quad B(d)^{rr} = (1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$$

and similarly for $B(d)^{sr}$, $B(t)^{rr}$, and $B(t)^{sr}$. Spatial versions of forward linkages follow the same kind of pattern. These are summarized in Table 12.4.

Examples of applications for single countries can be found in, among others, Blair and Miller (1990) and Shao and Miller (1990) for the US economy, Dietzenbacher (1992) for the Netherlands, Pan and Liu (2005) and Okamoto (2005) for China. The

**Table 12.4** Summary of Spatial/Sectoral Linkage Measures (Two-Region Example)

| Spatial/Sectoral Linkages | | | |
|---|---|---|---|
| Backward | | Forward | |
| Direct | Total | Direct | Total |
| $\mathbf{b}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}$,<br>$\mathbf{b}(d)^{sr} = \mathbf{i}'\mathbf{A}^{sr}$ | $\mathbf{b}(t)^{rr} = \mathbf{i}'\mathbf{L}^{rr}$,<br>$\mathbf{b}(t)^{sr} = \mathbf{i}'\mathbf{L}^{sr}$ | $\mathbf{f}(d)^{rr} = \mathbf{B}^{rr}\mathbf{i}$,<br>$\mathbf{f}(d)^{rs} = \mathbf{B}^{rs}\mathbf{i}$ | $\mathbf{f}(t)^{rr} = \mathbf{G}^{rr}\mathbf{i}$,<br>$\mathbf{f}(t)^{rs} = \mathbf{G}^{rs}\mathbf{i}$ |
| Normalizations include division of each direct element by $BL(d)_j^r$ [or each total element by $BL(t)_j^r$] or by $x_j$;<br>e.g., $\bar{\mathbf{b}}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}\langle\mathbf{b}(d)^r\rangle^{-1}$ or $\bar{\mathbf{b}}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}(\hat{\mathbf{x}})^{-1}$ | | Normalizations include division of each direct element by $FL(d)_j^r$ [or each total element by $FL(t)_j^r$] or by $x_j$;<br>e.g., $\bar{\mathbf{f}}(d)^{rr} = \langle\mathbf{f}(d)^r\rangle^{-1}\mathbf{B}^{rr}\mathbf{i}$ or $\bar{\mathbf{f}}(d)^{rr} = (\hat{\mathbf{x}})^{-1}\mathbf{B}^{rr}\mathbf{i}$ | |
| Spatial Linkages | | | |
| Backward | | Forward | |
| Direct | Total | Direct | Total |
| $B(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$,<br>$B(d)^{sr} = \mathbf{i}'\mathbf{A}^{sr}\mathbf{i}$ | $B(t)^{rr} = \mathbf{i}'\mathbf{L}^{rr}\mathbf{i}$,<br>$B(t)^{sr} = \mathbf{i}'\mathbf{L}^{sr}\mathbf{i}$ | $F(d)^{rr} = \mathbf{i}'\mathbf{B}^{rr}\mathbf{i}$,<br>$F(d)^{rs} = \mathbf{i}'\mathbf{B}^{rs}\mathbf{i}$ | $F(t)^{rr} = \mathbf{i}'\mathbf{G}^{rr}\mathbf{i}$,<br>$F(t)^{rs} = \mathbf{i}'\mathbf{G}^{rs}\mathbf{i}$ |
| | Normalizations include division of each element by $n$ or by $\mathbf{i}'\mathbf{x}$;<br>e.g., $\bar{B}(d)^{rr} = (1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$ or $\bar{B}(d)^{rr} = (1/\mathbf{i}'\mathbf{x})\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$ | | |

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 562.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=596

study by Chow, Lee and Ong (2006) for Singapore was based on 144-sector input–output data for 1990, 1995, and 2000. The authors chose to calculate (total) forward linkages using row sums of **A** rather than **B**.

With the emergence of international input–output data sets (for example, for the European Union and for Asia-Pacific economies, as described in section 8.8), these spatial measures are pertinent to questions of international economic connections and dependencies and their evolution over time. Illustrative applications here include Dietzenbacher and van der Linden (1997) for the countries of the European Community (see below) and Wu and Chen (2006) on backward linkages Taiwan ← Japan, Korea←Japan, China←Japan, and also Japan←Taiwan, Japan←Korea, Japan←China, in 1985, 1990, 1995, and 2000.

Alternative definitions for interregional economic connections are grounded in the notions of interregional feedbacks and spillovers (Miller and Blair, 1988; see also Chapters 3 and 6, above). These are closely related to the "hypothetical extraction" method. It provides a general framework for linkage analysis, and we turn to it next.

### *12.2.6 Hypothetical Extraction*

The objective of the hypothetical extraction approach is to quantify how much the total output of an $n$-sector economy would change (decrease) if a particular sector, say the $j$th, were removed from that economy. Initially, this was modeled in an input–output context by deleting row and column $j$ from the **A** matrix.[17] Using $\bar{\mathbf{A}}_{(j)}$ for the $(n-1)\times(n-1)$ matrix without sector $j$ and $\bar{\mathbf{f}}_{(j)}$ for the correspondingly reduced final-demand vector, output in the "reduced" economy is found as $\bar{\mathbf{x}}_{(j)} = [\mathbf{I}-\bar{\mathbf{A}}_{(j)}]^{-1}\bar{\mathbf{f}}_{(j)}$.[18] (Instead of physically deleting row and column $j$ in **A** and element $j$ in **f**, they can simply be replaced by zeros.) In the full $n$-sector model, output is $\mathbf{x}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$, so $T_j = \mathbf{i}'\mathbf{x}-\mathbf{i}'\bar{\mathbf{x}}_{(j)}$ is one aggregate measure of the economy's loss (decrease in value of gross output) if sector $j$ disappears – as such, it is a measure of the "importance" or *total linkage* of sector $j$. It has been argued that the first term, $\mathbf{i}'\mathbf{x}$, should not include the (original) output $x_j$. If $x_j$ is omitted, $(\mathbf{i}'\mathbf{x}-x_j)-\mathbf{i}'\bar{\mathbf{x}}$ would measure $j$'s importance to the remaining sectors in the economy. In either case, normalization through division by total gross output ($\mathbf{i}'\mathbf{x}$) and multiplication by 100 produces an estimate of the percentage decrease in total economic activity; $\bar{T}_j = 100[(\mathbf{i}'\mathbf{x}-\mathbf{i}'\bar{\mathbf{x}}_{(j)})/\mathbf{i}'\mathbf{x}]$. (Web Appendix 12W.1 presents hypothetical extractions in detail in the context of partitioned matrix versions of the Leontief and Ghosh models, along with citations to much more of the relevant literature.)

The hypothetical extraction approach has also been used to measure backward and forward linkage components separately (for example, in Dietzenbacher and van der Linden, 1997). Taking inspiration from the discussion of backward linkage in section 12.2.1 and forward linkage in section 12.2.2, **A** is used for this backward measure and **B** is used for the forward measure.

[17] The original idea seems to have appeared in Paelinck, de Caevel and Degueldre, 1965 (in French) or Strassert, 1968 (in German). The first discussion in English known to us is in Schultz (1976, 1977; the latter paper is a longer version of the former).

[18] We use $x_{(j)}$ to distinguish this linkage measure from $x_j$ which is the $j$th element in **x**.

*Backward Linkage* Assume that sector $j$ buys no intermediate inputs from any production sector; that is, remove sector $j$'s backward linkages. This is done by replacing column $j$ in $\mathbf{A}$ by a column of zeros. Denote this new matrix $\bar{\mathbf{A}}_{(cj)}$. (We used $\bar{\mathbf{A}}_{(j)}$ above to denote $\mathbf{A}$ with both row *and* column $j$ deleted; now we need the "$c$" to indicate that it is *column* $j$ only that is gone.) Then $\bar{\mathbf{x}}_{(cj)} = [\mathbf{I} - \bar{\mathbf{A}}_{(cj)}]^{-1}\mathbf{f}$ and $\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(cj)}$ is one measure of (aggregate) backward linkage for sector $j$. If more detail is of interest, each element $x_i - \bar{x}_{(cj)i}$ in $\mathbf{x} - \bar{\mathbf{x}}_{(cj)}$ can be viewed as the backward dependence of sector $j$ on sector $i$. Normalizations are possible and often used. For example, $[x_i - \bar{x}_{(cj)i}]/x_j$, puts this measure on a per-unit-of-output basis, or $100 \times [x_i - \bar{x}_{(cj)i}]/x_j$ to avoid relatively small numbers.

*Forward Linkage* A parallel to eliminating column $j$ in $\mathbf{A}$ as a way of identifying backward linkages might appear to be the elimination of row $j$ in $\mathbf{A}$ in order to quantify forward linkages. But the discussion in section 12.2.2 suggests that forward linkages of sector $j$ can be more appropriately identified through elimination of that sector's intermediate sales in the $\mathbf{B}$ matrix. That is, replace *row* $j$ of the output coefficients matrix by a row of zeros. Denote this matrix as $\bar{\mathbf{B}}_{(rj)}$. Then $\mathbf{x}' = \mathbf{v}'(\mathbf{I} - \mathbf{B})^{-1}$ and $\bar{\mathbf{x}}'_{(rj)} = \mathbf{v}'[\mathbf{I} - \bar{\mathbf{B}}_{(rj)}]^{-1}$ indicate pre- and post-extraction outputs, and $\mathbf{x}'\mathbf{i} - [\bar{\mathbf{x}}'_{(rj)}]\mathbf{i}$ is an aggregate measure of sector $j$'s forward linkage. Again, each element in $\mathbf{x}' - \bar{\mathbf{x}}'_{(rj)}$ is an indication of $j$'s dependence on sector $i$ as an intermediate output buyer, and normalizations are usual, as in $[x_i - \bar{x}_{(rj)i}]/x_j$ or $100 \times [x_i - \bar{x}_{(rj)i}]/x_j$.

In Table 12.5 we summarize the main hypothetical extraction results. [We (arbitrarily) use $B(t)_j$ and $F(t)_j$ instead of $BL(t)_j$ and $FL(t)_j$ to indicate results from the hypothetical extraction approach and to distinguish them from the linkage measures in (12.24) and (12.29).] The interested reader can work through the exercise of extending these extraction possibilities to the spatial context in which a *region* is hypothetically extracted from its many-region system in order to assess that region's backward, forward and/or total spatial linkages to the rest of that system. (This would amount to filling in boxes in the style of Table 12.5 for "Region $r$ Backward or Forward Linkage" and "Region $r$ Total Linkage.")

When linkages are being measured in order to make comparisons of the structure of production between countries, the underlying coefficients matrices, whether $\mathbf{A}$ or $\mathbf{B}$, should be derived from *total* interindustry transactions data – that is, a particular $z_{ij}$ should include good $i$ used by sector $j$, whether good $i$ comes from domestic producers or is imported. This is simply because interest is concentrated on how things are made in various economies, not on where the inputs come from. On the other hand, if linkages are being used to define "key" sectors in a particular economy, then the $\mathbf{A}$ or $\mathbf{B}$ matrices should be derived from a flow matrix that includes only domestically supplied inputs, since it is the impact on the domestic economy that is of interest. In studying the economies of less developed countries, it has been suggested (Bulmer-Thomas, 1982, p. 196) that "linkage analysis for LDCs is probably the most common use to which their input–output tables have been put."

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 564.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=598

**Table 12.5** Hypothetical Extraction Linkages

| Sector $j$ Backward or Forward Linkage | |
|---|---|
| Total Backward | Total Forward |
| $B(t)_j = \mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(cj)}$ where<br>$\bar{\mathbf{x}}_{(cj)} = [\mathbf{I} - \bar{\mathbf{A}}_{(cj)}]^{-1}\mathbf{f}$ | $F(t)_j = \mathbf{x}'\mathbf{i} - [\bar{\mathbf{x}}'_{(rj)}]\mathbf{i}$ where<br>$\bar{\mathbf{x}}'_{(rj)} = \mathbf{v}'[\mathbf{I} - \bar{\mathbf{B}}_{(rj)}]^{-1}$ |
| Normalizations include (1) division of each element by $x_j$, or<br>(2) division by $\sum_{j=1}^{n} x_j$ to create the percentage decrease in total output,<br>$\bar{B}(t)_j = 100\{[\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(cj)}]/\mathbf{i}'\mathbf{x}\}$ and $\bar{F}(t)_j = 100\{[\mathbf{x}'\mathbf{i} - \bar{\mathbf{x}}'_{(rj)}\mathbf{i}]/\mathbf{x}'\mathbf{i}\}$<br>or (3) values relative to the average, $\bar{\bar{B}}(t)_j = n\bar{B}(t)_j/\mathbf{i}'\bar{B}(t)_j$ and $\bar{\bar{F}}(t)_j = n\bar{F}(t)_j/\mathbf{i}'\bar{F}(t)_j$ | |
| Sector $j$ Total Linkage | |
| $T_j = \mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)}$ or $(\mathbf{i}'\mathbf{x} - x_j) - \mathbf{i}'\bar{\mathbf{x}}_{(j)}$<br>where $\bar{\mathbf{x}}_{(j)} = [\mathbf{I} - \bar{\mathbf{A}}_{(j)}]^{-1}\mathbf{f}_{(j)}$ | |
| Normalize to create percentage decrease in total output,<br>$\bar{T}_j = 100\{[\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)}]/\mathbf{i}'\mathbf{x}\}$ or $\bar{\bar{T}}_j = 100\{[\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)} - x_j]/\mathbf{i}'\mathbf{x}\}$<br>or to indicate values relative to the average,<br>$\bar{T}_j = n\bar{T}_j / \sum_{j=1}^{n} \bar{T}_j$ or $\bar{\bar{T}}_j = n\bar{\bar{T}}_j / \sum_{j=1}^{n} \bar{\bar{T}}_j$ | |

An applied study that uses both backward and forward linkages (as in sections 12.2.1 and 12.2.2) as well as incorporating a spatial dimension (section 12.2.4) along with hypothetical extraction (section 12.2.5) is found in Dietzenbacher and van der Linden (1997). This application is based on 1980 intercountry data for seven countries and 17 sectors of the European Community. It begins with backward and forward sectoral linkages for each country. These are then split into domestic and external linkages (the other six countries). Summations over all sectors give average (backward or forward) linkage of each country to each other. Then hypothetical extraction is applied to each country – for example, removal of sector $j$ in Germany leads to an output reduction in Germany and an output reduction in the other six countries. These are translated into percentages (domestic vs. intercountry), giving a measure of each country's importance in the European Community economic system. Also sums (and averages) over all sectors in each country generate linkages between each pair of countries.

### 12.2.7 Illustration Using US Data

Results for the US 2003 seven-sector tables (Chapter 2) are collected in Table 12.6. As expected, the elements in $\mathbf{b}(t)$ are the total output multipliers that were found in Chapter 6. Normalized backward linkages (either direct or total) identify the sectors with the three strongest (above average) normalized backward linkages ($\overline{BL} > 1$) as

**Table 12.6** Linkage Results, US 2003 Data

| Sector | $\mathbf{b}(d)$ | $\mathbf{b}(t)$ | $\mathbf{f}(d)$ | $\mathbf{f}(t)$ | $\bar{\mathbf{b}}(d)$ | $\bar{\mathbf{b}}(t)$ | $\bar{\mathbf{f}}(d)$ | $\bar{\mathbf{f}}(t)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.51 | 1.92 | 0.75 | 2.46 | 1.26 | 1.13 | 1.78 | 1.42 |
| 2 | 0.37 | 1.61 | 0.64 | 2.11 | 0.90 | 0.95 | 1.51 | 1.21 |
| 3 | 0.42 | 1.72 | 0.13 | 1.20 | 1.03 | 1.02 | 0.30 | 0.69 |
| 4 | 0.53 | 1.93 | 0.46 | 1.76 | 1.30 | 1.14 | 1.08 | 1.01 |
| 5 | 0.30 | 1.49 | 0.38 | 1.63 | 0.74 | 0.88 | 0.90 | 0.94 |
| 6 | 0.37 | 1.61 | 0.45 | 1.74 | 0.91 | 0.95 | 1.05 | 1.00 |
| 7 | 0.36 | 1.60 | 0.16 | 1.27 | 0.88 | 0.94 | 0.38 | 0.73 |

**Table 12.7** Classification of Linkage Results, US 2003 Data

| | | Direct $[\bar{\mathbf{f}}(d)]$ or Total $[\bar{\mathbf{f}}(t)]$ Forward Linkage | |
|---|---|---|---|
| | | Low ( $<1$) | High ( $>1$) |
| Direct $[\bar{\mathbf{b}}(d)]$ or Total $[\bar{\mathbf{b}}(t)]$ Backward Linkage | Low ( $<1$) | 5 (Trade, Transp., Utilities), 7 (Other) | 2 (Mining), 6 (Services) |
| | High ( $>1$) | 3 (Construction) | 1 (Agriculture), 4 (Manufacturing) |

**Table 12.8** Hypothetical Extraction Results, US 2003 Data

| Sector | $\bar{B}(t)_j$ | $\bar{F}(t)_j$ | $\bar{T}_j$ | $\bar{\bar{T}}_j$ |
|---|---|---|---|---|
| 1 | 1.02 | 1.61 | 2.12 | 0.73 |
| 2 | 0.69 | 1.27 | 1.84 | 0.61 |
| 3 | 3.87 | 1.06 | 17.78 | 12.39 |
| 4 | 13.59 | 11.20 | 30.02 | 10.30 |
| 5 | 6.47 | 8.42 | 25.51 | 13.03 |
| 6 | 19.95 | 24.44 | 59.69 | 13.38 |
| 7 | 6.64 | 2.98 | 25.85 | 14.38 |
| Sector | $\tilde{B}(t)_j$ | $\tilde{F}(t)_j$ | $\tilde{T}_j$ | $\tilde{\bar{T}}_j$ |
| 1 | 0.14 | 0.22 | 0.09 | 0.08 |
| 2 | 0.09 | 0.17 | 0.08 | 0.07 |
| 3 | 0.52 | 0.15 | 0.76 | 1.34 |
| 4 | 1.82 | 1.54 | 1.27 | 1.11 |
| 5 | 0.87 | 1.16 | 1.17 | 1.41 |
| 6 | 2.67 | 3.36 | 2.54 | 1.44 |
| 7 | 0.89 | 0.41 | 1.10 | 1.55 |

**Table 12.9** Classification of Hypothetical Extraction Results, US 2003 Data

| | | Total Forward Linkage $[\tilde{F}(t)_j]$ | |
|---|---|---|---|
| | | Low (<1) | High (>1) |
| Total Backward Linkage $[\tilde{B}(t)_j]$ | Low (<1) | 1 (Agriculture), 2 (Mining) 3 (Construction), 7 (Other) | 5 (Trade, Transp., Utilities) |
| | High (>1) | | 4 (Manufacturing), 6 (Services) |

(4) Manufacturing, (1) Agriculture, and (3) Construction, in that order. In the case of normalized forward linkages, the three largest (above average) are (1) Agriculture, (2) Mining, and (4) Manufacturing, in that order. These results are arranged in Table 12.7.

Hypothetical extraction results for the seven sectors are shown in Tables 12.8 and 12.9. With so few sectors it is perhaps not surprising that the orderings are similar across these hypothetical extraction measures, although netting out $x_j$ does produce some changes in rankings. Sectors 6 (Services) and 4 (Manufacturing) are identified as the two "most important" to the economy – with $\tilde{B}(t)_j > 1$, $\tilde{F}(t)_j > 1$, $\tilde{T}_j > 1$, and $\tilde{\tilde{T}}_j > 1$. However, as the reader can verify, the four-way classifications differ considerably from those in Table 12.7.

## 12.3 Identifying Important Coefficients

There is a long history and an enormous amount of published work, both theoretical and empirical, on the impact (transmission, propagation) of errors or changes or uncertainty in basic input–output data on the model outcomes. This has appeared under a variety of titles ("probabilistic" or "stochastic" input–output, "error" analysis and "sensitivity" analysis, and so on). Examples in the "probabilistic" vein go back at least to Quandt (1958, 1959).[19] Approaches that investigate the impacts of discrete changes in one or more model components go back at least to the early 1950s (Dwyer and Waugh, 1953; Evans, 1954). It is beyond the scope of this book to explore all of this literature. (A brief review and a large set of pertinent references can be found in Lahr, 2001.) Instead, we concentrate on the concept of "important coefficients."

Early mathematical work on the notion of "important" coefficients ($ICs$) in an input–output model explored ways of identifying $a_{ij}$ coefficients that have a particularly strong influence on one or more elements in the model, usually on the associated Leontief inverse matrix and/or on one or more gross outputs – meaning that $\Delta a_{ij} \rightarrow$ a "large" $\Delta l_{rs}$ or that $\Delta a_{ij} \rightarrow$ a "large" $\Delta x_r$ for one or more $r$ and $s$. (We will explore below what

19 Also representative of this line of inquiry are Simonovits (1975), Lahiri (1983), West (1986), Jackson and West (1989), Roland-Holst (1989), Kop Jansen (1994), ten Raa (1995, Chapter 14; 2005, Chapter 14), or Dietzenbacher (1995, 2006) and the many additional publications cited in these references.

**Table 12.10** Number of Important Transactions in the 2000 China MRIO Model

| Criterion | Number of cells | Percentage of total number of cells |
|---|---|---|
| $> (\mathbf{i}'\mathbf{Zi}/n^2)$ | 4,715 | 8.19 |
| $> (10) \times (\mathbf{i}'\mathbf{Zi}/n^2)$ | 1,042 | 1.81 |
| $> (100) \times (\mathbf{i}'\mathbf{Zi}/n^2)$ | 84 | 0.15 |
| $> (200) \times (\mathbf{i}'\mathbf{Zi}/n^2)$ | 32 | 0.06 |

constitutes "large" and what measures of "change" are used in these investigations.) Identification of such coefficients can be helpful in deciding where to expend effort in obtaining superior information for updating or regionalizing a known input–output table using a hybrid model. And *IC*s contribute to some studies of key sectors and of what has come to be known as "fundamental economic structure." Jackson (1991) suggests that a distinction should be made between coefficient error (e.g., estimation error) and coefficient change (e.g., technological change).

In what follows, we examine the mathematical underpinnings of these approaches and then several kinds of studies – primarily influences on inverse elements and on gross outputs. Reviews of much of this work can be found in Xu and Madden (1991), Casler and Hadlock (1997) and Tarancón *et al.* (2008). There are many more published studies than we are able to cite. A large amount of work was done in Germany in the 1970s and 1980s and published in German, making it somewhat less accessible to a segment of the English-speaking audience – for example, Schintke (1979, 1984), Maaß (1980) and numerous references therein.

One very straightforward way to assess "importance" of individual cells in input–output data is simply to compare each transaction ($z_{ij}$) with the average transaction amount ($\mathbf{i}'\mathbf{Zi}/n^2$). This is done in Okamoto (2005) for the 2000 China multiregional input–output data (CMRIO) made up of eight regions with 30 sectors each – a total of 57,600 potential elements in $\mathbf{Z}$. Table 12.10 shows results for this particular data set (adapted from Okamoto, 2005, p. 141). A similar approach could also be used on the data in coefficients matrices ($\mathbf{A}$ or $\mathbf{B}$) or total requirements matrices ($\mathbf{L}$ or $\mathbf{G}$).

### *12.3.1 Mathematical Background*

These investigations build on early results in Sherman and Morrison (1949, 1950) and Woodbury (1950) – hereafter SMW – who studied how changes in elements in a (nonsingular) matrix were transmitted to changes in elements in the inverse of that matrix. (Basic results and additional details are presented in Appendix 12.1.) Given a nonsingular matrix, $\mathbf{M}$, and its inverse, $\mathbf{M}^{-1} = [\mu_{ij}]$, assume that one (or more) elements of $\mathbf{M}$ are changed, i.e., $m^*_{ij} = m_{ij} + \Delta m_{ij}$, producing $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$. SMW show how the elements of $(\mathbf{M}^*)^{-1} = [\mu^*_{ij}]$ can be found by "adjusting" the known elements $\mu_{ij}$. This is addressed by Sherman and Morrison (1950) for the case when only one element is changed, by Sherman and Morrison (1949) for changes in several elements in a given column or row and by Woodbury (1950) for changes in elements in several rows (or columns).[20]

For the simplest situation, when a single element $m_{ij}$ is changed (increased or decreased) by an amount $\Delta m_{ij}$, the value of the element in row $r$ and column $s$ of the new inverse is found to be

$$\mu_{rs}^{*} = \mu_{rs} - \frac{\mu_{ri}\mu_{js}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} \tag{12.30}$$

Building on this result, it is possible to trace the influence of a change (or "error") in an element of an $\mathbf{A}$ matrix – and hence in $(\mathbf{I} - \mathbf{A})$ – on the associated Leontief inverse, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. In this case, we begin with $\mathbf{A}^{*} = \mathbf{A} + \Delta\mathbf{A}$. Since our interest is in $\mathbf{L}^{*} = (\mathbf{I} - \mathbf{A}^{*})^{-1}$, the parallel to $\mathbf{M}^{*} = \mathbf{M} + \Delta\mathbf{M}$ is

$$(\mathbf{I} - \mathbf{A}^{*}) = [\mathbf{I} - (\mathbf{A} + \Delta\mathbf{A})] = (\mathbf{I} - \mathbf{A}) + (-\Delta\mathbf{A})$$

For individual elements in the new inverse, $l_{rs}^{*}$, the result in (12.30) becomes

$$l_{rs}^{*} = l_{rs} + \frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} \tag{12.31}$$

[This is (A12.1.4) in Appendix 12.1.] Notice that the signs are reversed from those in (12.30) because of the way in which $\Delta\mathbf{A}$ enters the expression for $(\mathbf{I} - \mathbf{A}^{*})$.

### *12.3.2 Relative Sizes of Elements in the Leontief Inverse*

The following observations on Leontief inverse elements are relevant to the problem of identifying important coefficients. As we will see, they help to reduce the number of coefficients that need to be examined when ranking those elements for importance.

*Observation 1* From the power series approximation, it is clear that all on-diagonal elements in a Leontief inverse are larger than one. Also, it is virtually always observed in real-world Leontief inverse matrices that $l_{rs} < 1$ $(r \neq s)$ (off-diagonal elements are less than one);[21] thus $l_{ii} > 1 > l_{rs}$ for all $r \neq s$. This will be of use for the results below in (12.32).

[20] For much more detail on all of these results, see Miller (2000, Appendices 5.2 and 6.1).

[21] This is not to say that a counterexample cannot be constructed, but rather that they do not seem to occur in practice. For example

$$\mathbf{A} = \begin{bmatrix} 0.02 & 0.4 & 0.4 \\ 0.3 & 0.05 & 0.3 \\ 0.4 & 0.30 & 0.01 \end{bmatrix} \Rightarrow \mathbf{L} = \begin{bmatrix} 1.7767 & 1.0779 & 1.0445 \\ 0.8711 & 1.6925 & 0.8649 \\ 0.9818 & 0.9484 & 1.6942 \end{bmatrix}.$$

As an illustration, all the US Leontief inverses in Miller and Blair (1985, Appendix B), from 1947 through 1977, at both 23- and seven-sector levels of aggregation, exhibit the properties of Observation 1.

*Observation 2* In (12.33) and (12.42), it will be of interest to identify the largest of the ratios $l_{ri}l_{js}/l_{rs}$ for a given $i$ and $j$. Schnabl (2003, p. 497) reports on results in Maaß (1980, in German)

> Maaß's calculation showed that the maximum [of these ratios] is attained if $r = i$ and $s = j$ because then the main diagonal element of the inverse is involved *twice* and since the main diagonal element is usually the biggest one in a row or column this gives the maximum.

Thus $\underset{r,s=1,\ldots,n}{\text{Max}} \; l_{ri}l_{js}/l_{rs} = l_{ii}l_{jj}/l_{ij}$.

*Observation 3* Finally, $\underset{r=1,\ldots,n}{\text{Max}} \; l_{ri}/x_r = l_{ii}/x_i$. It is not at all obvious that this should be the case, since the sizes of sectors (as measured by their gross outputs) can vary greatly in real-world models. Nonetheless, it was observed in some early empirical observations and is proven to always be the case in Tarancón *et al.* (2008)[22]. This is useful for the results in (12.38).

### *12.3.3 "Inverse-Important" Coefficients*

For the remainder of this section, it will be useful to complicate the notation in order to be explicit about the element in **A** that is changed. From (12.31),

$$\Delta l_{rs(ij)} = l^*_{rs(ij)} - l_{rs} = \frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = l_{ri}l_{js}k^1_{(ij)} \tag{12.32}$$

where $k^1_{(ij)} = \Delta a_{ij}/(1 - l_{ji}\Delta a_{ij})$, a constant for a given $i$ and $j$, and $\mathbf{L}^*_{(ij)} = [l^*_{rs(ij)}]$ reminds us that the change is in $a_{ij}$. From Observation 1, above, $\Delta a_{ij}$ will exert the largest influence on $l_{ij}$ when $r = i$ and $s = j$, since then both elements multiplying $k^1_{(ij)}$ are larger than one [when $l_{ri}$ is $l_{ii}(>1)$ and $l_{js}$ is $l_{jj}\ (>1)$]. Similarly, next-largest influences will be felt in row $i$ or column $j$ of **L**, since then either $l_{ri} \to l_{ii} > 1$ or $l_{js} \to l_{jj} > 1$. In virtually all other cases (not row $i$ or column $j$) both elements of the product $l_{ri}l_{js}$ are less than one.

From (12.32), the expression for *relative* changes in Leontief inverse elements is

$$\frac{\Delta l_{rs(ij)}}{l_{rs}} = \frac{l_{ri}l_{js}\Delta a_{ij}}{l_{rs}(1 - l_{ji}\Delta a_{ij})} = \frac{l_{ri}l_{js}}{l_{rs}}k^1_{(ij)} \tag{12.33}$$

This is where Observation 2 becomes relevant. Since $\underset{r,s=1,\ldots,n}{Max} \; l_{ri}l_{js}/l_{rs} = l_{ii}l_{jj}/l_{ij}$, it is clear that, again, $\Delta a_{ij}$ will create the largest relative change on $l_{ij}$.

In addition, the elements $\Delta l_{rs(ij)}/l_{rs}$ in *column* $i$ and *row* $j$ of the matrix of relative changes will all be identical. In column $i$ (when $s = i$), $\Delta l_{ri(ij)}/l_{ri} = (l_{ri}l_{ji}/l_{ri})k^1_{(ij)} = l_{ji}k^1_{(ij)}$, and in row $j$ (when $r = j$), $\Delta l_{js(ij)}/l_{js} = (l_{ji}l_{js}/l_{js})k^1_{(ij)} = l_{ji}k^1_{(ij)} = \Delta l_{ri(ij)}/l_{ri}$.

[22] Sekulić (1968) observed this to be true for the Yugoslav economy in the early 1960s. Similar observations are made in Schintke (1979 and elsewhere) based on German data. See also results from US data in Table 12.12, below.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 570.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=604

Finally, the *percentage* changes are

$$p_{rs(ij)} = 100\left[\frac{\Delta l_{rs(ij)}}{l_{rs}}\right] = 100\left[\frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{1}{l_{rs}}\right] = 100\left[\frac{l_{ri}l_{js}}{l_{rs}}k^{1}_{(ij)}\right] \tag{12.34}$$

Again, $p_{ij(ij)}$ will be the largest percentage change caused by $\Delta a_{ij}$.

It has been suggested (for example, Hewings, 1981) that $a_{ij}$ may be viewed as "inverse-important" if, for a specified "threshold" amount of change in an inverse element, $\beta$, $p_{rs(ij)} \geq \beta$ for one or more $r$ and $s$, that is, if

$$p_{rs(ij)} = 100\left[\frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{1}{l_{rs}}\right] \geq \beta \tag{12.35}$$

Denote the percentage change in $a_{ij}$ by $\alpha$, so that $\Delta a_{ij} = [\alpha/100]\,a_{ij}$; then we have

$$\left[\frac{l_{ri}l_{js}\alpha a_{ij}}{100 - l_{ji}\alpha a_{ij}}\right]\left[\frac{100}{l_{rs}}\right] \geq \beta \tag{12.36}$$

for any $l_{rs}$ and a given $\alpha$ and $\beta$. For example, let $\alpha = 20$ and $\beta = 10$. This means that $a_{ij}$ will be considered inverse-important if a 20 percent change in its value generates a 10 percent or larger change in one or more elements in the Leontief inverse. The analyst must specify $\alpha$ and $\beta$, on the basis of the particular problem under study.

Given Observations 1 and 2, establishing inverse importance for *each* $a_{ij}$ in an $n$-sector **A** matrix requires only one application of (12.35) [or (12.36)] – for $r = i$ and $s = j$.[23] The virtue of the SMW method is that it finds this information about the inverse by working exclusively with known elements in **L** and avoiding direct calculation of the new inverse. This was the whole point of the formulation.

At present, however, finding inverses is not quite the task it was in 1950 when the SMW approach was developed; at least this is true for matrices that are not "too large." Then a straightforward alternative to applying (12.35) or (12.36) is to calculate directly the $\mathbf{L}^{*}_{(ij)}$ associated with each $\Delta a_{ij}$ and then find the corresponding matrix of percentage changes, $\mathbf{P}_{(ij)} = [p_{rs(ij)}] = 100\{[\mathbf{L}^{*}_{(ij)} - \mathbf{L}]\oslash\mathbf{L}\}$, where "$\oslash$" indicates element-by-element division.

This line of work, formulating the notion of inverse-important coefficients, was taken up initially in the early 1980s by Hewings, Jensen, West, and others. Examples are Jensen and West (1980), Hewings (1981), Hewings and Romanos (1981) and Hewings (1984). For hybrid (partial-survey) models, the idea is to identify coefficients (or sectors) for which additional information (survey, expert opinion) would be particularly useful. But of course identifying inverse importance implies that a relevant matrix of coefficients already exists to supply the elements in results like (12.35). For updating,

[23] If one wants not simply to establish inverse-importance, but also *extent* [that is, for a given $\Delta a_{ij}$, how many (and which) $p_{rs(ij)}$ exceed the $\beta$ threshold], then $[n^2 - (2n-1)]$ calculations like those in (12.35) or (12.36) would be needed – the $n^2$ inverse elements, $l_{rs}$, for a given $a_{ij}$, less those in row $j$ and column $i$ that are all identical – and these calculations must be made $n$ times, once for each of the $a_{ij}$. The "field of influence" approach (section 12.3.6) accomplishes this in one matrix operation.

there is a base matrix to be updated, and the premise is that important coefficients at time "$t$" will also be important at time "$t+1$." However, there is no hard evidence to support that argument. In fact, Hewings (1984, p. 325), cautioned that "…only 3 of the cells deemed inverse-important in 1963 [the Washington State 49-sector model] were similarly identified in 1967." For regional models there is often not an "earlier" regional table. In the context of estimating a coefficients table in a regional context, Boomsma and Oosterhaven (1992, p. 276, n. 3) observed:

> Here we have a typical "chicken or egg" problem. Without a regional table one cannot determine the inverse-important cells and without that information one cannot construct a decent regional table. Hence, we suggest use of the national table as second best information on inverse-importance.

### *12.3.4 Numerical Example*

We use the two-sector example closed with respect to households from section 2.5, namely[24]

$$\mathbf{A} = \begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$$

Consider $\Delta a_{12} = (0.2)a_{12}$ (that is, $\alpha = 20$); then $\mathbf{A}^* = \begin{bmatrix} .15 & .30 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$, and we can easily calculate $\mathbf{L}^*_{(12)}$ directly as

$$\mathbf{L}^*_{(12)} = \begin{bmatrix} 1.4021 & .5198 & .2926 \\ .5416 & 1.3846 & .6115 \\ .5853 & .5285 & 1.3060 \end{bmatrix}$$

Then

$$\mathbf{P}_{(12)} = \begin{bmatrix} 2.7080 & 22.2225 & 16.6345 \\ 2.7080 & 2.7080 & 2.7080 \\ 2.7080 & 8.0667 & 1.3521 \end{bmatrix}$$

As expected, with the change $\Delta a_{12}$, all elements in column 1 and row 2 are identical.

Further, in this illustration $l_{jj} > 1$ $(j = 1, \ldots, 3)$, $l_{ij} < 1$ $(i = 1, \ldots, 3; i \neq j)$ (Observation 1, above) and indeed the largest change caused by $\Delta a_{12}$ is in $l_{12}$ – here this is $p_{12(12)} = 22.2$ percent. If we specify $\beta = 10$ as our criterion in (12.36) for inverse-importance – namely when a change of 10 percent or more is experienced by at least one inverse coefficient – then we see that $a_{12}$ would be classified as inverse-important because $\Delta a_{12} = (0.2)a_{12}$ causes both $l_{12}$ and $l_{13}$ to be changed by more than 10 percent. In this small three-sector case, it is relatively easy to modify each element in $\mathbf{A}$,

24 In section 2.5 these matrices contained overbars to indicate a model closed with respect to households and to distinguish them from the earlier open model. At this point the overbars just get in the way of other notation and will be dropped.

in turn, by 20 percent, find the associated Leontief inverse and then the associated **P** matrix. (Readers are encouraged to do this, at least for several additional $a_{ij}$.) If we continue to use $\beta = 10$ in that series of calculations – for $\Delta a_{11} = (0.2)a_{11}, \ldots, \Delta a_{33} = (0.2)a_{33}$ – we will identify $a_{21}, a_{23}, a_{31}$, and $a_{32}$ also as important.[25] (Higher values of $\beta$ serve to raise the bar on eligibility for importance. For example, with $\beta = 20$, only $a_{12}$ and $a_{23}$ would be labeled important.)

"Importance" could be identified in many other ways, for example, with respect to changes in output multipliers – as in $100\{[\mathbf{i}'\mathbf{L}^*_{(ij)} - \mathbf{i}'\mathbf{L}] \oslash \mathbf{i}'\mathbf{L}\} = 100\{[\mathbf{i}'\Delta\mathbf{L}_{(ij)}] \oslash \mathbf{i}'\mathbf{L}\}$. If $\beta$ now refers to percentage change in multipliers, only $a_{23}$ is found to be important with $\beta = 10$; however, at $\beta = 5$ the same five coefficients as above are identified.

As noted, the point of the SMW result is that these percentage changes in inverse coefficients can be found without knowing the new inverse at all. Continuing with $\Delta a_{12} = (0.2)a_{12}$, consider the percentage change in $l_{13}$ [$p_{13(12)}$ in $\mathbf{P}_{(12)}$, above]. Using (12.35) with $i = 1, j = 2, r = 1, s = 3$, and $\Delta a_{12} = 0.05$, we have

$$p_{13(12)} = \left[\frac{l_{11}l_{23}\Delta a_{12}}{1 - l_{21}\Delta a_{12}}\right]\left[\frac{100}{l_{13}}\right] = \left[\frac{(1.3651)(.5954)(0.05)}{1 - (.5273)(0.05)}\right]\left[\frac{100}{(.2509)}\right] = 16.6359$$

Except for rounding (and the number of significant digits carried in the inversion programs used to find $\mathbf{L}$ and $\mathbf{L}^*_{(12)}$), this corresponds to the $p_{13(12)}$ found above. Any other value in $\mathbf{P}_{(12)}$ could be found in the same way.

The designation of inverse-importance depends crucially on the choice of $\alpha$ and $\beta$. In a study of several of the Washington State 49-sector tables, Hewings (1984) used $\alpha = 30$ and $\beta = 20$. Out of $49 \times 49 = 2401$ direct input coefficients, between 24 and 42 (1.0–1.7 percent) were judged inverse-important. In a similar study in Sri Lanka (also Hewings, 1984), 3.5 percent were found important in a 12-sector model (apparently using the same $\alpha$ and $\beta$). There were interesting although not surprising variations in a two-region Sri Lanka interregional input–output model between intraregional and interregional coefficients (now with a $24 \times 24$ matrix); 3.3 percent of the (possible) 288 intraregional coefficients were important and 0.9 percent of those 288 in the interregional matrices were important. In a similar study (Hewings and Romanos, 1981) using a 22-sector model for the rural Evros region in Greece, 18 of 484 possible coefficients (3.7 percent) were important – only here, because of the less-developed nature of the economy, the critical values used were $\alpha = 20$ and $\beta = 1$. With those same values, a 22-sector model for the Greek national economy had 38 important coefficients (7.9 percent).

### 12.3.5 Impacts on Gross Outputs

Early applications of these ideas to the impact of coefficient change on gross outputs are found in Sekulić (1968) and Jílek (1971).[26] In matrix terms, $\Delta\mathbf{x}_{(ij)} = \mathbf{x}^*_{(ij)} - \mathbf{x} =$

[25] In each of the eight cases the largest change caused by that $\Delta a_{ij}$ was in the associated $l_{ij}$, as expected, but four of those were below the $\beta = 10$ threshold.

[26] Sekulić (1968) and later Jílek (1971) attribute the approach to E. B. Yershof who contributed a chapter of a 1965 Moscow publication on planning (in Russian). The bases of the approach are in the work of SMW, some 15 years earlier.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 573.
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$\mathbf{L}^*_{(ij)}\mathbf{f} - \mathbf{L}\mathbf{f} = \Delta\mathbf{L}_{(ij)}\mathbf{f}$. From (12.31), we see that row $r$ of $\Delta\mathbf{L}_{(ij)}$ is

$$\begin{bmatrix}\Delta l_{r1(ij)} & \cdots & \Delta l_{rn(ij)}\end{bmatrix} = \frac{l_{ri}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\begin{bmatrix}l_{j1} & \cdots & l_{jn}\end{bmatrix}$$

and therefore

$$\Delta x_{r(ij)} = \begin{bmatrix}\Delta l_{r1(ij)} & \cdots & \Delta l_{rn(ij)}\end{bmatrix}\begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix} = \left[\frac{l_{ri}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\begin{bmatrix}l_{j1} & \cdots & l_{jn}\end{bmatrix}\begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix}$$

But since $\begin{bmatrix}l_{j1} & \cdots & l_{jn}\end{bmatrix}\begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix} = x_j$, this is just

$$\Delta x_{r(ij)} = \frac{l_{ri}x_j\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = l_{ri}k^2_{(ij)} \tag{12.37}$$

where $k^2_{(ij)} = x_j\Delta a_{ij}/(1 - l_{ji}\Delta a_{ij})$. Compared with the expression for $\Delta l_{rs(ij)}$ in (12.32), $l_{js}$ has been replaced on the right-hand side by $x_j$. Again, from Observation 1, $l_{ii} > l_{ri}$ (for $r = 1, \ldots, n$; $r \neq i$), so (12.37) indicates that the largest gross output change from $\Delta a_{ij}$ will be in sector $i$ (that is, when $r = i$).[27]

The *relative* change in $x_r$ is then

$$\frac{\Delta x_{r(ij)}}{x_r} = \frac{l_{ri}x_j\Delta a_{ij}}{x_r(1 - l_{ji}\Delta a_{ij})} = \left[\frac{l_{ri}}{x_r}\right]k^2_{(ij)} \tag{12.38}$$

Here the largest *relative* change in gross output for a given $\Delta a_{ij}$ will be in sector $s$ for which $l_{si}/x_s = \underset{r=1,\ldots,n}{\text{Max}}\ (l_{ri}/x_r)$, and from Observation 3, this will be for sector $i$. The interested reader can easily show this to be true for the numerical example, above. Table 12.11 presents the same calculations for the 2003 US seven-sector data from Chapter 2, showing that the largest ratios (in bold) are on the main diagonal.

Finally, multiplication in (12.38) by 100 creates a *percentage* change,

$$100\left[\frac{\Delta x_{r(ij)}}{x_r}\right] = 100\left[\frac{\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] \tag{12.39}$$

Table 12.12 contains these percentages for $x_1$, $x_2$, and $x_3$ from the hypothetical example as a result of $\Delta a_{ij} = (0.2)a_{ij}$ for all nine direct input coefficients ($i, j = 1, 2, 3$).

As expected, for any $\Delta a_{ij}$, the largest changes are found in $x_i$; in the row for $i = 1$, this means $\Delta x_1 > \Delta x_2$ and $\Delta x_1 > \Delta x_3$, and so on in the rows for $i = 2$ and $i = 3$. Also, with $\alpha = 20$, if the criterion for "importance" is that one or more *outputs* changes by

[27] If $x_j$ is small relative to other outputs, then a large $\Delta x_j$ may not have much economy-wide importance. There have been attempts to take this aspect of relative output size into account, but we do not consider this level of detail. The interested reader might speculate on how this could be done.

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**Table 12.11** $(l_{ri}/x_r) \times 10^6$ for the 2003 US Seven-Sector Model

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $r=1$ | **4.5868** | 0.0211 | 0.0477 | 0.2093 | 0.0136 | 0.0253 | 0.0263 |
| $r=2$ | 0.0381 | **4.4187** | 0.0501 | 0.1408 | 0.0794 | 0.0136 | 0.0301 |
| $r=3$ | 0.0071 | 0.0032 | **0.9449** | 0.0060 | 0.0061 | 0.0105 | 0.0235 |
| $r=4$ | 0.0589 | 0.0306 | 0.0672 | **0.3449** | 0.0178 | 0.0220 | 0.0324 |
| $r=5$ | 0.0523 | 0.0297 | 0.0480 | 0.0547 | **0.3811** | 0.0209 | 0.0298 |
| $r=6$ | 0.0261 | 0.0321 | 0.0295 | 0.0319 | 0.0297 | **0.1544** | 0.0343 |
| $r=7$ | 0.0107 | 0.0105 | 0.0102 | 0.0162 | 0.0124 | 0.0131 | **0.4566** |

**Table 12.12** Percentage Change in **x** Resulting from $\Delta a_{ij} = (0.2)a_{ij}$

| | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|
| $i=1$ | $\begin{bmatrix} 4.27 \\ 0.82 \\ 1.78 \end{bmatrix}$ | $\begin{bmatrix} \mathbf{14.02} \\ 2.71 \\ 5.85 \end{bmatrix}$ | $\begin{bmatrix} 1.37 \\ 0.27 \\ 0.57 \end{bmatrix}$ |
| $i=2$ | $\begin{bmatrix} 1.73 \\ 2.74 \\ 1.99 \end{bmatrix}$ | $\begin{bmatrix} 0.86 \\ 1.37 \\ 0.99 \end{bmatrix}$ | $\begin{bmatrix} 3.54 \\ 5.61 \\ 4.07 \end{bmatrix}$ |
| $i=3$ | $\begin{bmatrix} 1.53 \\ 1.81 \\ 7.85 \end{bmatrix}$ | $\begin{bmatrix} 2.59 \\ 3.07 \\ \mathbf{13.28} \end{bmatrix}$ | $\begin{bmatrix} 0.25 \\ 0.30 \\ 1.31 \end{bmatrix}$ |

$\beta = 10$ percent, then $a_{12}$ and $a_{32}$ would be labeled most and second-most important. These percentage changes are indicated in bold in the table. [The interested reader might speculate on why it is not surprising that coefficients judged important by the criterion in (12.35) are likely to be tagged as important by the criterion in (12.39).]

Again, the "importance" of any $a_{ij}$ could be defined in terms of the impact of *relative* or *percentage* changes in $a_{ij}$ on the associated relative or percentage changes in each $x_r$. Using $\gamma_r$ for the (user-specified) threshold on percentage changes in $x_r$,

$$\left[\frac{100\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] \geq \gamma_r$$

[Compare (12.35).] Again, with $\Delta a_{ij} = [\alpha/100]\, a_{ij}$, we have

$$\left[\frac{\alpha a_{ij}}{1 - l_{ji}\left[\frac{\alpha}{100}\right] a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] = \left[\frac{100\alpha a_{ij}}{100 - l_{ji}\alpha a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] \geq \gamma_r$$

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Cambridge, , GBR: Cambridge University Press, 2009. p 575.
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**Table 12.13** Upper Threshold on $\Delta a_{ij}/a_{ij}$ for $\gamma = 1$ Percent

| | $j = 1$ | $j = 2$ | $j = 3$ |
|---|---|---|---|
| $i = 1$ | 4.90 | 1.47 | 14.71 |
| $i = 2$ | 7.46 | 14.92 | 3.73 |
| $i = 3$ | 2.58 | 1.55 | 15.50 |

Much of the empirical work in this area is based on a rearrangement of (12.39). Putting $\Delta a_{ij}$ on the left and converting to *relative* change in $a_{ij}$, we have

$$\frac{\Delta a_{ij}}{a_{ij}} = \frac{\Delta x_{r(ij)}/x_r}{a_{ij}[(l_{ji}\Delta x_{r(ij)}/x_r) + (l_{ri}x_j/x_r)]}$$

Define an allowable error limit, $\gamma$, for *all* sectors $r$ which is just fulfilled by positive relative deviations $\Delta a_{ij}/a_{ij}$. This is often called a "tolerable limit, TL" and hence the name "tolerable limits approach." As is frequently done, let $\gamma = 100(\Delta x_{r(ij)}/x_r) = 1$ percent; then in percentage terms

$$\frac{\Delta a_{ij}}{a_{ij}} = \frac{100\Delta x_{r(ij)}/x_r}{a_{ij}[(l_{ji}100\Delta x_{r(ij)}/x_r) + 100(l_{ri}x_j/x_r)]} = \frac{1}{a_{ij}[l_{ji} + 100(l_{ri}/x_r)x_j]}$$

Expressed in this way, we see that the larger the denominator on the right-hand side, the smaller $\Delta a_{ij}/a_{ij}$. So the upper threshold on $\Delta a_{ij}/a_{ij}$ will be determined by $\underset{r=1,\ldots,n}{\text{Max}}\ l_{ri}/x_r$,

$$\frac{\Delta a_{ij}}{a_{ij}} \le \frac{1}{a_{ij}[l_{ji} + 100 \underset{r=1,\ldots,n}{\text{Max}}(l_{ri}/x_r)x_j]}$$

As noted (Observation 3) $\underset{r=1,\ldots,n}{\text{Max}}\ l_{ri}/x_r = l_{ii}/x_i$, so

$$\frac{\Delta a_{ij}}{a_{ij}} \le \frac{1}{a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j]} \tag{12.40}$$

establishes an *upper limit* on the relative change in $a_{ij}$ that assures that *no* gross output will be changed by more than one percent.[28] The smaller $\Delta a_{ij}/a_{ij}$, the more important the coefficient $a_{ij}$. Table 12.13 shows the right-hand sides of (12.40) for our small numerical example.

From the upper-left element in the table, we learn that $a_{11}$ could change by as much as 4.9 percent before any output would be changed by more than one percent. Similarly,

[28] This can be found in Sekulić (1968). Forssell (1989, p. 431) describes it as a measure "developed by Mäenpää (1981)" but it seems to have been suggested much earlier. The Jugoslav journal in which the Sekulić paper appeared may not be well known, but the paper was also presented at the Fourth International Conference on input–output Techniques in Geneva in 1968.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 576.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=610

**Table 12.14** Average Values in US Total Requirements Matrices

| Number of Sectors | $\mathbf{i}'\hat{\mathbf{L}}\mathbf{i}/n$ | $\mathbf{i}'\check{\mathbf{L}}\mathbf{i}/(n^2-n)$ |
|---|---|---|
| $n = 7$ | 1.1739 | 0.0868 |
| $n = 16$ | 1.1290 | 0.0429 |
| $n = 61$ | 1.1113 | 0.0133 |

$a_{12}$ (1.47) is identified as the most important coefficient (smallest value in Table 12.13), followed by $a_{32}$ (1.55), $a_{31}$(2.58) and so on. While perhaps not of much interest, we would also conclude that $a_{33}$ is least important, since it could change by as much as 15.5 percent before any gross output would be changed by more than one percent. [Since the result in (12.40) comes directly from the result in (12.39), it should not be surprising that the importance rankings of the nine coefficients in our numerical example that are shown in Tables 12.12 and 12.13 are exactly the same.][29]

The denominator on the right in (12.40),

$$a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j]$$

has been described as a measure of the "degree of importance" of $a_{ij}$ (for example, by Schintke and Stäglin, 1984). In real-world applications it turns out that $l_{ji} \ll 100(l_{ii}/x_i)x_j$, especially for relatively disaggregated input–output models, again because of Observation 1 ($l_{ii} > 1 > l_{ij}$). In fact, there are usually quite large differences between the $l_{ii}$ and the $l_{ij}$. For example, average values of on-diagonal elements (in $\hat{\mathbf{L}}$) and off-diagonal elements (in $\check{\mathbf{L}}$) in Leontief inverses for 2003 US input–output data are shown in Table 12.14.

This suggests that, for any given $a_{ij}$ and irrespective of $x_i$ and $x_j$,[30] the first term can be ignored and the measure can be approximated as

$$a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j] \approx 100a_{ij}(l_{ii}/x_i)x_j$$

Using $b_{ij} = z_{ij}/x_i = a_{ij}x_j/x_i$ (the usual "output coefficient" from the Ghosh model) this has also been expressed as

$$a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j] \approx 100b_{ij}l_{ii}$$

[29] Empirical examples identifying important coefficients for a variety of tolerable limits can be found in Aroche-Reyes (1996, 2002) for Mexico (1970, 1980) in the first case and for Mexico (1971, 1990), Canada (1972, 1990) and the US (1971, 1990) in the second.

[30] Of course one could generate counter examples with very large $x_i$ and very small $x_j$ so that $l_{ji} > 100(l_{ii}/x_i)x_j$. The point is that this does not seem to happen in real-world applications.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 577.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=611

### 12.3.6 Fields of Influence

In a number of articles, Sonis and Hewings and their colleagues have developed and applied the concept of a "field of influence" associated with each coefficient in an **A** matrix.[31] This is essentially an extension of the Sherman–Morrison approach that generates in one operation the entire matrix of changes in the Leontief inverse associated with a given change in a particular $a_{ij}$. Recall that $\Delta l_{rs(ij)}$ is related to $\Delta a_{ij}$ through

$$\Delta l_{rs(ij)} = l^*_{rs(ij)} - l_{rs} = \frac{l_{ri} l_{js} \Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = l_{ri} l_{js} k^1_{(ij)}$$

[This is (12.32), above.] Finding all the $n^2$ elements in the $n \times n$ matrix $\Delta \mathbf{L}_{(ij)} = [\Delta l_{rs(ij)}]$ would require $[n^2 - (2n-1)]$ operations, as we saw above (footnote 21). Instead, Sonis and Hewings propose an efficient alternative.

Let column $i$ and row $j$ of **L** be denoted $\mathbf{L}_{\cdot i} = \begin{bmatrix} l_{1i} \\ l_{2i} \\ \vdots \\ l_{ni} \end{bmatrix}$ and $\mathbf{L}_{j\cdot} = \begin{bmatrix} l_{j1} & l_{j2} & \cdots & l_{jn} \end{bmatrix}$.

Then the first order (direct) field of influence of the incremental change $\Delta a_{ij}$ is defined by Sonis and Hewings as the matrix[32]

$$\mathbf{F}[i,j] = \mathbf{L}_{\cdot i}\mathbf{L}_{j\cdot} = \begin{bmatrix} l_{1i} \\ l_{2i} \\ \vdots \\ l_{ni} \end{bmatrix} \begin{bmatrix} l_{j1} & l_{j2} & \cdots & l_{jn} \end{bmatrix} = \begin{bmatrix} l_{1i}l_{j1} & l_{1i}l_{j2} & \cdots & l_{1i}l_{jn} \\ l_{2i}l_{j1} & l_{2i}l_{j2} & \cdots & l_{2i}l_{jn} \\ \vdots & \vdots & & \vdots \\ l_{ni}l_{j1} & l_{ni}l_{j2} & \cdots & l_{ni}l_{jn} \end{bmatrix}$$

Thus, $\mathbf{F}[i,j] = [l_{ri}l_{js}]$ for $r, s = 1, \ldots, n$ is the expanded version of the product $l_{ri}l_{js}$ on the right-hand side of (12.32), and the *matrix* showing the change in each element of **L** caused by $\Delta a_{ij}$ is just $\Delta \mathbf{L}_{(ij)} = \mathbf{F}[i,j]k^1_{(ij)}$. Therefore

$$\mathbf{L}^*_{(ij)} = \mathbf{L} + \Delta\mathbf{L}_{(ij)} = \mathbf{L} + [(\Delta a_{ij})/(1 - l_{ji}\Delta a_{ij})]\mathbf{F}[i,j] = \mathbf{L} + \mathbf{F}[i,j]k^1_{(ij)}$$

Since $k^1_{(ij)}$ is a constant for any specific $\Delta a_{ij}$, corresponding elements of $\Delta\mathbf{L}_{(ij)}$ and $\mathbf{F}[i,j]$ are proportional and will have the same ordering – for example, largest to smallest.

In the numerical example, $\mathbf{L}_{\cdot i} = \mathbf{L}_{\cdot 1} = \begin{bmatrix} 1.3651 \\ 0.5273 \\ 0.5698 \end{bmatrix}$ and

[31] The publications are numerous, going back at least to Sonis and Hewings (1989). A fairly compact statement can be found in Sonis and Hewings (1992) and an application (to the Chicago economy) is presented in Okuyama *et al.* (2002).

[32] Sonis and Hewings used $\mathbf{F}\begin{pmatrix} i \\ j \end{pmatrix}$ to indicate a field of influence in early publications; later (for example, Sonis and Hewings, 1999) this became $\mathbf{F}[i,j]$.

$\mathbf{L}_{j\cdot} = \mathbf{L}_{2\cdot} = \begin{bmatrix} 0.5273 & 1.3481 & 0.5954 \end{bmatrix}$ so

$$\mathbf{F}[1,2] = \mathbf{L}_{\cdot 1}\mathbf{L}_{2\cdot} = \begin{bmatrix} 0.7198 & 1.8402 & 0.8127 \\ 0.2781 & 0.7109 & 0.3139 \\ 0.3005 & 0.7682 & 0.3393 \end{bmatrix}$$

Further, $\Delta a_{12} = 0.05$ and $k^1_{(12)} = [(\Delta a_{12})/(1 - l_{21}\Delta a_{12})] = 0.0514$ so

$$\Delta\mathbf{L}_{(12)} = \mathbf{F}[1,2](0.0514) = \begin{bmatrix} 0.0370 & 0.0945 & 0.0417 \\ 0.0143 & 0.0365 & 0.0161 \\ 0.0154 & 0.0395 & 0.0174 \end{bmatrix}$$

and it is easily verified that $\mathbf{L}^*_{(12)} = \mathbf{L} + \Delta\mathbf{L}_{(12)}$.

Sonis and Hewings suggest that inverse-important coefficients can be identified by comparing their fields of influence.[33] The problem is how to reduce the $n^2$ pieces of information in each $\mathbf{F}[i,j]$ in order to make comparisons across the $\Delta a_{ij}$.[34] The norms of these matrices offer one possible compact measure; the trouble is that there are many different definitions of a matrix norm. Among those that they mention (Sonis and Hewings, 1992, p. 147) are

$\|\mathbf{F}\| = \max\limits_{ij} |f_{ij}|$ (largest individual element)[35]

$\|\mathbf{F}\| = \sum\limits_{ij} |f_{ij}|$ (sum of all elements)

$\|\mathbf{F}\| = \left[\sum\limits_{ij} |f_{ij}|\right]^{1/2}$

In Chapter 2 we used a largest column sum norm; $\|\mathbf{F}\| = \max\limits_{j} \sum\limits_{i} |f_{ij}|$. Further,

> [t]he choice of norm $\|\mathbf{F}\|$ is the basis of the construction of the rank-size sequence of the elements $a_{ij}$ of the matrix $\mathbf{A}$ according to the numerical sizes of the norms $\|\mathbf{F}[i,j]\|$. The decision or cutting rule must be formulated in such a way that only a relatively small number of the elements of the rank-size sequence will comprise the set of inverse-important coefficients. (p. 147).

Returning to our numerical example, we generated fields of influence for each of the nine coefficients in $\mathbf{A}$ using $\Delta a_{ij} = (0.2)a_{ij}$ – namely, $\Delta a_{11} = (0.2)a_{11}$, then $\Delta a_{12} = (0.2)a_{12}$, and so on. Tables 12.15 and 12.16 present two summary measures (norms) from these nine $\mathbf{F}[i,j]$ matrices. Table 12.15 contains the column sums, and the $\|\mathbf{F}\| =$

[33] In other publications they also propose higher-order fields of influence when two or more coefficients change (with associated mathematical representations that are much more complicated), and they also use some of these concepts to characterize the fundamental structures of economies and provide alternative kinds of model decompositions.

[34] Generally, for comparability, each $a_{ij}$ is changed by the same percentage, $\alpha$, so that $\Delta a_{ij} = (\alpha/100)a_{ij}$ for all $i,j$. In presenting applications identifying important coefficients Sonis and Hewings (1992) do not specify either their choice of norm or their coefficient alteration mechanism.

[35] There is no need to generate the entire field of influence matrix if one is then going to summarize the information by using the $\max\limits_{ij}|f_{ij}|$ norm of that matrix. We know that the largest $\Delta l_{rs(ij)}$ is $\Delta l_{ij(ij)}$ and that $f_{rs(ij)}$ is proportional to $\Delta l_{rs(ij)}$ so this can be found using the Sherman–Morrison results in (12.32) for $\Delta l_{ij(ij)}$ only.

**Table 12.15** Column Sums of $|\mathbf{F}[i,j]|$ for Numerical Example

| $a_{ij}$ | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|
| $i=1$ | $[3.3612\quad 1.0471\quad 0.6178]$ | $[1.2984\quad 3.3193\quad 1.4659]$ | $[1.4031\quad 1.2042\quad 3.1727]$ |
| $i=2$ | $[3.0884\quad 0.9621\quad 0.5676]$ | $[1.1930\quad 3.0499\quad 1.3469]$ | $[1.2892\quad 1.1064\quad 2.9152]$ |
| $i=3$ | $[2.9142\quad 0.9078\quad 0.5356]$ | $[1.1257\quad 2.8779\quad 1.2710]$ | $[1.2165\quad 1.0440\quad 2.7508]$ |

**Table 12.16** Sum of all Elements in $|\mathbf{F}[i,j]|$ $\left(\|\mathbf{F}\| = \sum_{ij} |f_{ij}|\right)$

| $a_{ij}$ | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|
| $i=1$ | 5.0261 | 6.0837 | 5.7800 |
| $i=2$ | 4.6181 | 5.5898 | 5.3108 |
| $i=3$ | 4.3577 | 5.2746 | 5.0113 |

$\max_j \sum_i |f_{ij}|$ norm is obvious by inspection in each case. Table 12.16 contains the $\|\mathbf{F}\| = \sum_{ij} |f_{ij}|$ norm for the nine coefficients.

### 12.3.7 Additional Measures of Coefficient Importance

*Converting Output to Employment, Income, etc.* As noted many times earlier in this book, gross outputs may not ultimately be the most important measure of economic impact. Gross output requirements can be translated into employment (for example, person-years) using employment coefficients (for example, person-hours per dollar's worth of each sector's output). If these coefficients are denoted $\mathbf{e}_c$ and total employment in each sector is represented by $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$, then $\Delta\boldsymbol{\varepsilon} = \hat{\mathbf{e}}_c \Delta\mathbf{x}$ converts changes in outputs to changes in employment. For example, from (12.36),

$$\Delta\varepsilon_r = (e_c)_r \Delta x_{r(ij)} = \frac{(e_c)_r l_{ri} x_j \Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = (e_c)_r l_{ri} k^2_{(ij)} \quad \text{where} \quad k^2_{(ij)} = \frac{x_j \Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}$$

The largest employment impact of $\Delta a_{ij}$ will thus be in the sector with the largest $(e_c)_r l_{ri}$, and this is no longer assured to be sector $i$. Numerous other conversions are also possible – for example, to changes in income, value added, energy use, environmental impacts, and so forth.[36]

*Elasticity Coefficient Analysis* Several authors have suggested a variation of the measure of relative change that parallels the concept of *elasticity* in economics (see section 6.6), namely the relative change in $l_{rs(ij)}$ divided by the relative change in $a_{ij}$

$$\eta_{l_{rs(ij)}} = \frac{\frac{\Delta l_{rs(ij)}}{l_{rs}}}{\frac{\Delta a_{ij}}{a_{ij}}} = \frac{\frac{\Delta l_{rs(ij)}}{\Delta a_{ij}}}{\frac{l_{rs}}{a_{ij}}} = \left(\frac{\Delta l_{rs(ij)}}{\Delta a_{ij}}\right)\left(\frac{a_{ij}}{l_{rs}}\right) \tag{12.41}$$

From (12.33), this is

$$\eta_{l_{rs(ij)}} = \frac{l_{ri} l_{js} a_{ij}}{l_{rs}(1 - l_{ji}\Delta a_{ij})} = \frac{l_{ri} l_{js}}{l_{rs}} k^3_{(ij)} \tag{12.42}$$

where $k^3_{(ij)} = a_{ij}/(1 - l_{ji}\Delta a_{ij})$. Notice that this differs from the expression for $\Delta l_{rs(ij)}/l_{rs}$ in (12.33) only in that $\Delta a_{ij}$ has been replaced by $a_{ij}$ in the numerator. For any $a_{ij}$, there will be $n^2$ of these elasticities. Then Maaß (1980; cited in Schnabl, 2003) proposed the maximum of these elasticities as another measure of the importance of $a_{ij}$ – $\text{Max}_{rs}(\eta_{l_{rs(ij)}})$. From Observation 2, again, it is clear that

$$\text{Max}_{rs}(\eta_{l_{rs(ij)}}) = \frac{l_{ii} l_{jj} a_{ij}}{l_{ij}(1 - l_{ji}\Delta a_{ij})}$$

So, as noted by Schnabl, this elasticity analysis generates the same results as the important coefficient analysis above.

Replacement of $\Delta l_{rs(ij)}/l_{rs}$ in the numerator in (12.41) by $\Delta x_{r(ij)}/x_r$ will lead to an expression for the elasticity of gross output with respect to $\Delta a_{ij}$. And, just as gross output impacts can be translated into employment, income, value-added, etc. effects, these variations too can be converted to elasticity measures.

*Relative Changes in All Gross Outputs* A straightforward error measure that takes into account changes in *all* outputs is

$$E_{(ij)} = \mathbf{i}' \left|\Delta \mathbf{x}_{(ij)}\right| = \sum_{k=1}^{n} \left|\Delta x_{k(ij)}\right|$$

Or, to take account of the relative sizes of the sectors, Siebe (1996) suggests

$$\text{SUM}_{(ij)} = \sum_{k=1}^{n} \left|\Delta x_{k(ij)}/x_k\right|$$

[36] Tarancón *et al.* (2008) discuss in some detail the identification of important coefficients using alternative measures of economic welfare.

as a measure of importance of each coefficient, $a_{ij}$. As with previous measures, this could be transformed into an aggregate effect on employment, income, value-added, etc.

*Impacts of Changes in more than One Element of the* A *Matrix* Assessing the importance of each $a_{ij}$ relative to all the others is carried out using one or more of the one-at-a-time approaches that we explored above. There has also been considerable work on the impacts of simultaneous changes (errors) in many or all $a_{ij}$ coefficients. Indeed Sherman and Morrison (1949) considered cases with more than one change, but concentrated in a single row (or column). This was also explored in many publications by Schintke (1979 and elsewhere) and Schintke and Stäglin (1984 and elsewhere). Since this is somewhat peripheral to our "important coefficient" interests, we confine some of the background and results to Web Appendix 12W.2.

## 12.4 Summary

Initially in this chapter we explored the supply-side (Ghosh) model with both its early and later interpretations, in terms of quantity and price models, respectively. Various approaches to measuring linkages in an input–output system were the topic of section 12.2. Early approaches identified backward and forward linkages through appropriate row and column sums of the Leontief and Ghosh coefficient matrices (**A** and **B**) or their counterpart inverses, **L** and **G**. An alternative and more comprehensive view of linkage measurement grew out of the notion of hypothetical extraction, which can be implemented for backward, forward or total linkage measures. A detailed classification of hypothetical extraction possibilities is presented in Web Appendix 12W.1. The final topic considered in this chapter is the problem of how to define (conceptually) and identify (mathematically) "important" coefficients in an input–output system. Many approaches have been suggested. A major reason for interest in this topic is that it helps to identify where one might concentrate resources when trying to improve (for example, update) an input–output model's data base. Some historical background and details on this issue are relegated to Appendix 12.1 and Web Appendix 12W.2.

## Appendix 12.1 The Sherman–Morrison–Woodbury Formulation

### *A12.1.1 Introduction*

Given a nonsingular matrix, **M**, and its inverse, suppose that one or more elements of **M** are changed, producing $\mathbf{M}^*$. The question is: can we find $(\mathbf{M}^*)^{-1} = [\mu^*_{ij}]$ by "adjusting" $\mathbf{M}^{-1} = [\mu_{ij}]$, which is already known? This is addressed by Sherman and Morrison (1949, 1950) for the case in which only one element is changed and by Woodbury (1950) for the case in which more than one element is changed. The answer

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 582.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=616

is "yes," and the adjustment is relatively simple.[37] (Hereafter we will refer to the "SMW" results.)

Here is an illustration for the case of a change in one element only (Miller, 2000, pp. 281–286). Given

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 3 & 7 & 1 \end{bmatrix} \text{ and } \mathbf{M}^{-1} = \begin{bmatrix} 3.5 & -0.5 & -0.5 \\ -1.3333 & 0.1667 & 0.3333 \\ -1.1667 & 0.3333 & 0.1667 \end{bmatrix}$$

consider an $\mathbf{M}^*$ that differs from $\mathbf{M}$ only in that 3 has been added to $m_{23}$, changing it from a 6 to a 9. Let $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$ where, in this case, $\Delta\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. For later reference, we can easily find

$$(\mathbf{M}^*)^{-1} = \begin{bmatrix} 2.625 & -0.25 & -0.375 \\ -1.0417 & 0.0833 & 0.2917 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix}$$

The idea is to find an alternative to the direct computation of $(\mathbf{M}^*)^{-1}$, making use only of $\mathbf{M}^{-1}$ and of the size of the change (here $\Delta m_{23} = 3$).[38]

The heart of the procedure is contained in two matrices (for this example, these are vectors). Let $\mathbf{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$; then $\Delta\mathbf{M} = \mathbf{CR}$. The trick is to let $\mathbf{C}$ be the $i$th column of an identity matrix (the same size as $\mathbf{M}$), where $i$ identifies the *row* in $\mathbf{M}$ in which the change occurs, and where $\mathbf{R}$ is an appropriately sized null row vector with the $j$th element replaced by $\Delta m_{ij}$. The fundamental result is

$$(\mathbf{M}^*)^{-1} = \mathbf{M}^{-1} - \Delta\mathbf{M}^{-1} = \mathbf{M}^{-1} - \frac{(\mathbf{M}^{-1}\mathbf{C})(\mathbf{R}\mathbf{M}^{-1})}{(1 + \mathbf{R}\mathbf{M}^{-1}\mathbf{C})} \tag{A12.1.1}$$

This is not as complex as it might appear. The numerator of $\Delta\mathbf{M}^{-1}$ is the product of a column vector $\mathbf{M}^{-1}\mathbf{C}$ and a row vector $\mathbf{R}\mathbf{M}^{-1}$ and the denominator is simply a scalar.[39]

The expression for an individual element in $(\mathbf{M}^*)^{-1}$ follows directly from (A12.1.1). For a matrix $\mathbf{M}$ in which element $m_{ij}$ is changed (increased or decreased) by $\Delta m_{ij}$, the value of the element in row $r$ and column $s$ of the new inverse, $\mu^*_{rs}$, is

$$\mu^*_{rs} = \mu_{rs} - \frac{\mu_{ri}\mu_{js}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} \tag{A12.1.2}$$

[37] Henderson and Searle (1981) is an important reference on inverses of sums of matrices that seems generally ignored in the input–output literature. It includes at least six different variations on the SMW results and an extensive set of references.

[38] If changes in each of several $a_{ij}$ are to be examined, it is helpful to use the notation $\mathbf{M}^*_{ij}$ in order to identify the specific case under consideration.

[39] A similar result can be derived with the roles of $\mathbf{R}$ and $\mathbf{C}$ interchanged (see Miller, 2000, Appendix 5.2).

The new elements in column $i$ and row $j$ of $(\mathbf{M}^*)^{-1}$ will be strictly proportional to the corresponding elements in $\mathbf{M}^{-1}$. For column $i$, when $s = i$,

$$\mu_{ri}^* = \mu_{ri} - \frac{\mu_{ri}\mu_{ji}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} = \frac{\mu_{ri} + \mu_{ri}\mu_{ji}\Delta m_{ij} - \mu_{ri}\mu_{ji}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} = \mu_{ri}k_{ij}$$

where $k_{ij} = 1/(1 + \mu_{ji}\Delta m_{ij})$ is a constant for a given $\Delta m_{ij}$, and exactly similar algebra shows that when $r = j$, $\mu_{js}^* = \mu_{js}k_{ij}$.

For the numerical example,

$$\mathbf{M}^{-1}\mathbf{C} = \begin{bmatrix} -0.5 \\ 0.1667 \\ 0.3333 \end{bmatrix}, \quad \mathbf{RM}^{-1} = \begin{bmatrix} -3.5 & 1 & 0.5 \end{bmatrix} \text{ and } \mathbf{RM}^{-1}\mathbf{C} = 1$$

so that, from (A12.1.1),

$$\Delta\mathbf{M}^{-1} = (0.5)\begin{bmatrix} 1.75 & -0.5 & -0.25 \\ -0.5833 & 0.1667 & 0.0833 \\ -1.1667 & 0.3333 & 0.1667 \end{bmatrix} = \begin{bmatrix} .875 & -0.25 & -0.125 \\ -0.2917 & 0.0833 & 0.0417 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix}$$

and

$$\begin{aligned}(\mathbf{M}^*)^{-1} &= \mathbf{M}^{-1} - \Delta\mathbf{M}^{-1} \\ &= \begin{bmatrix} 3.5 & -0.5 & -0.5 \\ -1.3333 & 0.1667 & 0.3333 \\ -1.1667 & 0.3333 & 0.1667 \end{bmatrix} - \begin{bmatrix} .875 & -0.25 & -0.125 \\ -0.2917 & 0.0833 & 0.0417 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix} \\ &= \begin{bmatrix} 2.625 & -0.25 & -0.375 \\ -1.0417 & 0.0833 & 0.2917 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix}\end{aligned}$$

This is exactly the inverse that was found directly earlier in this Appendix. The reader can easily check the results in (A12.1.2) for any of the elements in $(\mathbf{M}^*)^{-1}$.

The (obvious) point is that a change (here an increase of 50 percent) in the value of just one element in $\mathbf{M}$ leads to changes in *all* elements in $\mathbf{M}^{-1}$. Note that some changes are increases, as with $\mu_{12}$ (and three other elements), and some are decreases, as with $\mu_{11}$ (and four other elements). Absolute values of the percentage changes can be found as[40] $|p_{ij}| = 100\left|(\mu_{ij}^* - \mu_{ij})/\mu_{ij}\right|$, or

$$|\mathbf{P}| = 100\left|[(\mathbf{M}^*)^{-1} - \mathbf{M}^{-1}] \oslash \mathbf{M}^{-1}\right|$$

where "$\oslash$" indicates element-by-element division. Here

$$|\mathbf{P}| = \begin{bmatrix} 25 & 50 & 25 \\ 21.875 & 50 & 12.5 \\ 50 & 50 & 50 \end{bmatrix}$$

[40] Frequently the changes are expressed as $(\mu_{ij} - \mu_{ij}^*)/\mu_{ij}$. This simply reverses signs. If absolute values are used, it makes no difference.

As expected for this example with a change in $m_{23}$, the elements in column 2 and row 3 of $(\mathbf{M}^*)^{-1}$are proportional to the corresponding elements in $\mathbf{M}^{-1}$, and hence the percentage changes are all the same.[41]

### *A12.1.2 Application to Leontief Inverses*

The relevance to input–output models is that one can investigate the influence of changes (or "errors") in one or more elements of an $\mathbf{A}$ matrix on the associated Leontief inverse, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. Here we begin with $\mathbf{A}^* = \mathbf{A} + \Delta\mathbf{A}$ but since our interest is in $\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}$, the parallel to $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$ is

$$(\mathbf{I} - \mathbf{A}^*) = [\mathbf{I} - (\mathbf{A} + \Delta\mathbf{A})] = (\mathbf{I} - \mathbf{A}) + (-\Delta\mathbf{A})$$

and the result in (A12.1.1) becomes

$$\mathbf{L}^* = \mathbf{L} + \frac{(\mathbf{LC})(\mathbf{RL})}{1 - \mathbf{RLC}} \tag{A12.1.3}$$

Notice that negative and positive signs are interchanged, compared to (A12.1.1).

In terms of an individual element in the new inverse, $l^*_{rs}$, the parallel to (A12.1.2) for a change $\Delta a_{ij}$ is (with notation to remind us of which element in $\mathbf{A}$ is changed)

$$l^*_{rs(ij)} = l_{rs} + \frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} \tag{A12.1.4}$$

Again, note the changes in signs, this time compared to (A12.1.2). Define percentage differences in Leontief inverse elements as $\Delta l_{rs(ij)} = (l^*_{rs(ij)} - l_{rs})/l_{rs}$; then

$$100\left[\frac{\Delta l_{rs(ij)}}{l_{rs}}\right] = 100\left[\frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{1}{l_{rs}}\right] \tag{A12.1.5}$$

As before, all elements in row $j$ and in column $i$ of the matrix of absolute percentage differences will be the same.

## Problems

12.1 The centrally planned economy of Czaria is involved in its planning for the next fiscal year. The technical coefficients and total industry outputs for Czaria are given below:

a. Compute the output inverse for this economy.

b. If next year's value-added inputs for agriculture, mining, military manufactured products, and civilian manufacturing in Czaria are projected to be \$4,558 million, \$5,665 million, \$2,050 million and \$5,079 million, respectively, compute the projected GDP for Czaria next year.

[41] The fact that all these changes are 50 percent (the same as the increase in $m_{23}$) is a coincidence of this example only. Moreover, some of the changes are 50 percent increases ($\mu_{12}$ and $\mu_{31}$) and some are 50 percent decreases ($\mu_{22}$, $\mu_{32}$, and $\mu_{33}$).

| | 1 | 2 | 3 | 4 | Total Output |
|---|---|---|---|---|---|
| 1. Agriculture | 0.168 | 0.155 | 0.213 | 0.212 | 12,000 |
| 2. Mining | 0.194 | 0.193 | 0.168 | 0.115 | 15,000 |
| 3. Military Manufacturing | 0.105 | 0.025 | 0.126 | 0.124 | 12,000 |
| 4. Civilian Manufacturing | 0.178 | 0.101 | 0.219 | 0.186 | 16,000 |

c. Compute the new total gross production for each economic sector. Note that this is the "old view" of the Ghosh model as described in section 12.1.1.

12.2 Consider a case where $\mathbf{Z} = \begin{bmatrix} 13 & 75 & 45 \\ 53 & 21 & 48 \\ 67 & 68 & 93 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 130 \\ 150 \\ 220 \end{bmatrix}$ for base year.

a. If final demands for the next year are projected to be $\mathbf{f}^1 = \begin{bmatrix} 200 \\ 300 \\ 500 \end{bmatrix}$ and the change in interindustry transactions is expected to be $\Delta\mathbf{Z} = \begin{bmatrix} 0 & 5 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ what is the mean absolute percentage difference (MAPD) between the output coefficients for the base year and next year?

b. Now compute MAPD between the corresponding output inverses.

12.3 For input–output transactions matrix of $\mathbf{Z} = \begin{bmatrix} 384 & 520 & 831 \\ 35 & 54 & 530 \\ 672 & 8 & 380 \end{bmatrix}$ and total outputs of $\mathbf{x} = \begin{bmatrix} 2500 \\ 1200 \\ 3000 \end{bmatrix}$ for a base year, if additional growth in value added for the next year is projected to result in $\mathbf{v}^{new} = \begin{bmatrix} 2000 \\ 1000 \\ 1500 \end{bmatrix}$, what are the price changes of output for the three industries for the new year relative to the base year?

12.4 For the economy shown in problem 12.3, compute the value-added coefficients for next year using the supply model. Compute **L** and show that the Leontief price model from Chapter 2 produces the same relative price changes of industrial output for the new year relative to the base year as found in problem 12.3.

12.5 Consider the case of $\mathbf{Z} = \begin{bmatrix} 418 & 687 & 589 & 931 \\ 847 & 527 & 92 & 654 \\ 416 & 702 & 911 & 763 \\ 263 & 48 & 737 & 329 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 2000 \\ 3000 \\ 2500 \\ 1500 \end{bmatrix}$.

a. Compute the direct and total backward linkages.

b. Compute the direct and total forward linkages.

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http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=620

12.6 Consider the three-region IRIO table for Japan given in Table A4.1.1. Using the measure of spatial backward linkage of $B(d)^{rr} = (1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$ (and analogous measures for direct forward and total backward and forward linkage), which of the three regions is the "least backward linked" to the other regions and, similarly, which region is the least "forward linked"?

12.7 Consider the 2005 US input–output table provided in Appendix B.

a. If the agriculture sector were hypothetically extracted from the economy, what would be the decrease in total output of the economy?

b. Which of the sectors would create the largest decrease in total output if it were hypothetically extracted?

12.8 Consider an economy with $\mathbf{Z} = \begin{bmatrix} 8 & 64 & 89 \\ 28 & 44 & 77 \\ 48 & 24 & 28 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 300 \\ 250 \\ 200 \end{bmatrix}$. Examine element $a_{13}$ for "inverse importance" if the criteria are:

a. $\alpha = 30$ and $\beta = 5$ – that is, if a 30 percent change in $a_{13}$ generates a 5 percent change in one or more elements in the associated Leontief inverse.

b. $\alpha = 20$ and $\beta = 10$.

c. $\alpha = 10$ and $\beta = 10$.

This illustrates the sensitivity of the results to the values of $\alpha$ and $\beta$ specified by the analyst.

12.9 Create a supply-driven model for the US economy for 2005 using the data that are presented in Appendix B. Determine the sensitivity of the national economy to an interruption in a scarce-factor input – for example, a strike – in one of the sectors.

12.10 Using the input–output data for the United States presented in Appendix B, find both the direct and the total forward and backward linkages for the sectors in the US economy and examine how these linkages may have changed over time.

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