13 Structural Decomposition, Mixed and Dynamic Models

13.1 Structural Decomposition Analysis

When there are two or more sets of input–output data for an economy, analysts are often interested in trying to disaggregate the total amount of change in some aspect of that economy into contributions made by its various components. For example, the total change in gross outputs between two periods could be broken down into that part associated with changes in technology (as reflected, initially, in the changes in the Leontief inverse for the economy over the period) and that part related to changes in final demand over the period.

At the next level, the total change in the Leontief inverse matrix could be disaggregated into a part that is associated with changes in technology within each sector (as reflected in changes in the direct input coefficients matrix) and that part associated with changes in product mix within each sector. Similarly, the change in final demand could be further disaggregated into a part that reflects changes in the overall level of final demand and a part that captures changes in the composition of final demand. And there are numerous additional options – for example, there is no need to use only two contributing factors; changes in employment, value added, energy use, etc. may be of more economic interest than changes in gross outputs; and so on. For a general overview of this literature, see Rose and Casler (1996) or Dietzenbacher and Los (1997, 1998). Two early empirical examples of this kind of work can be found in Feldman, McClain and Palmer (1987) for the USA and Skolka (1989) for Austria.[1]

13.1.1 *Initial Decompositions: Changes in Gross Outputs*

To get a general idea of the structural decomposition analysis (SDA) approach, we initially explore gross output changes. Assume that there are two time periods for which input–output data are available. Using superscripts 0 and 1 for the two different years (0 earlier than 1), our illustration of structural decomposition in an input–output

[1] Schumann (1994), expanding on Schumann (1990), argues for the superiority of semi-closed models (for example, to household consumption) in general but also claims that structural decomposition analyses with such models lead to inferior results because they isolate sources of structural change that are less clear cut and more complex.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 593.
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model focuses on the differences in the gross output vectors for those two years. As usual, gross outputs in year t, $\mathbf{x}^t (t = 0, 1)$, are found in an input–output system as

$$\mathbf{x}^1 = \mathbf{L}^1\mathbf{f}^1 \text{ and } \mathbf{x}^0 = \mathbf{L}^0\mathbf{f}^0 \tag{13.1}$$

where $\mathbf{f}^t$ = the vector of final demands in year t, and $\mathbf{L}^t = (\mathbf{I}-\mathbf{A}^t)^{-1}$. Then the observed change in gross outputs over the period is

$$\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0 = \mathbf{L}^1\mathbf{f}^1 - \mathbf{L}^0\mathbf{f}^0 \tag{13.2}$$

The task is to decompose the total change in outputs into changes in the various components – in (13.2) that would (at least initially) mean separation into changes in $\mathbf{L}$ $(\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0)$ and changes in $\mathbf{f}$ $(\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0)$.[2] In order to remove the influence of price changes, we assume that all data are expressed in prices for a common year.

A number of alternative expansions and rearrangements of the terms in (13.2) can be derived. For example, using only year-1 values for $\mathbf{L}$ and only year-0 values for $\mathbf{f}$ – replacing $\mathbf{L}^0$ with $(\mathbf{L}^1 - \Delta\mathbf{L})$ and $\mathbf{f}^1$ with $(\mathbf{f}^0 + \Delta\mathbf{f})$ in (13.2) – we have

$$\Delta\mathbf{x} = \mathbf{L}^1(\mathbf{f}^0 + \Delta\mathbf{f}) - (\mathbf{L}^1 - \Delta\mathbf{L})\mathbf{f}^0 = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f}) \tag{13.3}$$

This simple algebra produces a straightforward decomposition of the total change in gross outputs into (1) a part that is attributable to changes in technology, $\Delta\mathbf{L}$, in this case weighted by year-0 final demands $(\mathbf{f}^0)$, and (2) a part that reflects final-demand changes, $\Delta\mathbf{f}$, which are here weighted by year-1 technology $(\mathbf{L}^1)$.

Notice that each term on the right-hand side of (13.3) has a certain amount of intuitive appeal – for example, $(\Delta\mathbf{L})\mathbf{f}^0 = \mathbf{L}^1\mathbf{f}^0 - \mathbf{L}^0\mathbf{f}^0$. The first term quantifies the output that would be needed to satisfy old (year-0) demand with new (year-1) technology; the second term is, of course, the output needed to satisfy old demand with old technology. So the difference is one reasonable measure of the effect of technology change. And $\mathbf{L}^1(\Delta\mathbf{f})$ in (13.3) has a similar kind of interpretation.

Alternatively, using only year-0 values for $\mathbf{L}$ and only year-1 values for $\mathbf{f}$, which means replacing $\mathbf{L}^1$ with $(\mathbf{L}^0 + \Delta\mathbf{L})$ and $\mathbf{f}^0$ with $(\mathbf{f}^1 - \Delta\mathbf{f})$, (13.2) becomes

$$\Delta\mathbf{x} = (\mathbf{L}^0 + \Delta\mathbf{L})\mathbf{f}^1 - \mathbf{L}^0(\mathbf{f}^1 - \Delta\mathbf{f}) = (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f}) \tag{13.4}$$

In this case, the technology change contribution is weighted by year-1 final demands and the final-demand change contribution is weighted by year-0 technology.

These alternatives, in (13.3) and (13.4), are equally valid in the sense that both are "mathematically correct," given (13.2) and the definitions $\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0$ and $\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0$. Yet clearly the measures in (13.3) of the individual contributions from changed technology and from changed final demands will be different from those in (13.4), except in the totally uninteresting and implausible case where $\mathbf{L}^1 = \mathbf{L}^0$ and/or $\mathbf{f}^1 = \mathbf{f}^0$ –

[2] In section 7.2.1 we explored some of the most frequently used approaches to assessing overall structural change. One frequently used measure was to compare $\mathbf{x}^1 = \mathbf{L}^1\mathbf{f}^1$ with $\mathbf{L}^0\mathbf{f}^1$, the output that $\mathbf{f}^1$ would have generated with $\mathbf{L}^0$ technology.

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no change in technology or no change in demand (or no change in either) over the period. The results in (13.3) and (13.4) can be derived from (13.2) in another way. For example, adding and subtracting $\mathbf{L}^1\mathbf{f}^0$ to (13.2), and rearranging, gives (13.3). Similarly, adding and subtracting $\mathbf{L}^0\mathbf{f}^1$ (outputs needed if year-1 demands were satisfied using year-0 technology) to (13.2) gives (13.4), after rearrangement.

And there is more. Other expressions emerge if only year-0 or only year-1 values are used for weights on both change terms. If we use year-0 weights exclusively, so that $\mathbf{L}^1$ and $\mathbf{f}^1$ are replaced by $(\mathbf{L}^0 + \Delta\mathbf{L})$ and $(\mathbf{f}^0 + \Delta\mathbf{f})$, then (13.2) becomes

$$\Delta\mathbf{x} = (\mathbf{L}^0 + \Delta\mathbf{L})(\mathbf{f}^0 + \Delta\mathbf{f}) - \mathbf{L}^0\mathbf{f}^0 = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^0(\Delta\mathbf{f}) + (\Delta\mathbf{L})(\Delta\mathbf{f}) \qquad (13.5)$$

In this case, both technology and final-demand changes are weighted by year-0 values, but an additional ("interaction") term – $(\Delta\mathbf{L})(\Delta\mathbf{f})$ – has appeared. Unlike the first two terms in (13.5), this new interaction term does not have an intuitively appealing interpretation.[3]

Finally, using only year-1 weights means putting $\mathbf{L}^0 = (\mathbf{L}^1 - \Delta\mathbf{L})$ and $\mathbf{f}^0 = (\mathbf{f}^1 - \Delta\mathbf{f})$ into (13.2), which becomes

$$\Delta\mathbf{x} = \mathbf{L}^1\mathbf{f}^1 - (\mathbf{L}^1 - \Delta\mathbf{L})(\mathbf{f}^1 - \Delta\mathbf{f}) = (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{f}) - (\Delta\mathbf{L})(\Delta\mathbf{f}) \qquad (13.6)$$

again with the same interaction term, only this time it is subtracted rather than added.[4]

Various researchers have worked with one or more of these four alternatives. For example, Skolka (1989) presented the first three decompositions;[5] Rose and Chen (1991) work only with the expression in equation (13.5), although ultimately in an expanded form. Vaccara and Simon (1968) used the factorizations in (13.3) and (13.4), then averaged the two measures of final-demand change and the two measures of coefficient change. This is also the approach of Feldman, McClain and Palmer (1987), Miller and Shao (1994) and others. Dietzenbacher and Los (1998) examine a wide variety of possible decompositions and conclude that using an average of results from (13.3) and (13.4) is often an acceptable approach.[6]

We can view this as follows. Adding (13.3) and (13.4) gives

$$2\Delta\mathbf{x} = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f}) + (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f})$$

and so

$$\Delta\mathbf{x} = (1/2)\underbrace{(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change}} + (1/2)\underbrace{(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change}} \qquad (13.7)$$

3 Derivation of this result by adding and subtracting like terms in (13.2) is possible but more complicated. In fact, it requires that $\mathbf{L}^1\mathbf{f}^0$, $\mathbf{L}^0\mathbf{f}^1$, and $\mathbf{L}^0\mathbf{f}^0$ all be both added and subtracted and then (considerably) rearranged.

4 This result can be derived by adding and subtracting $\mathbf{L}^1\mathbf{f}^0$, $\mathbf{L}^0\mathbf{f}^1$, and $\mathbf{L}^1\mathbf{f}^1$ in (13.2) and (again) extensive algebraic rearrangement.

5 He also classifies much of the pre-1989 work in this area according to which version of the decomposition was used.

6 Not everyone would agree. Fromm (1968) discusses the index number issues that are involved in finding averages of measures with weights from different years. In terms of (13.3), the $(\Delta\mathbf{L})\mathbf{f}^0$ term is a kind of Laspeyres index (original year weights, in $\mathbf{f}^0$) and the $\mathbf{L}^1(\Delta\mathbf{f})$ term is a kind of Paasche index (terminal year weights in $\mathbf{L}^1$); in (13.4) the Laspeyres and Paasche terms are reversed. He suggests that averaging the two – (13.3) and (13.4) – gives a "... bastard measure of beginning- and end-point quantities and prices" (p. 65).

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[The average in (13.7) is the same as the average of the results in (13.5) and (13.6), as the reader can easily show.][7]

Numerical Example Here is a small numerical illustration of these decompositions. Let

$$\mathbf{Z}^0 = \begin{bmatrix} 10 & 20 & 25 \\ 15 & 5 & 30 \\ 30 & 40 & 5 \end{bmatrix}, \quad \mathbf{f}^0 = \begin{bmatrix} 45 \\ 30 \\ 25 \end{bmatrix}, \quad \mathbf{Z}^1 = \begin{bmatrix} 12 & 15 & 35 \\ 24 & 11 & 30 \\ 36 & 50 & 8 \end{bmatrix}, \quad \mathbf{f}^1 = \begin{bmatrix} 50 \\ 35 \\ 26 \end{bmatrix}$$

From $\mathbf{x}^0 = \mathbf{Z}^0\mathbf{i} + \mathbf{f}^0$ and $\mathbf{x}^1 = \mathbf{Z}^1\mathbf{i} + \mathbf{f}^1$, $\mathbf{L}^0$ and $\mathbf{L}^1$ are easily found, as are

$$\Delta\mathbf{L} = \begin{bmatrix} .0649 & -.0941 & .0320 \\ .1447 & .0607 & .0116 \\ .1448 & .0342 & .0586 \end{bmatrix}, \quad \Delta\mathbf{f} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} \text{ and } \quad \Delta\mathbf{x} = \begin{bmatrix} 12 \\ 20 \\ 20 \end{bmatrix}$$

The alternative decompositions of $\Delta\mathbf{x}$, for this example, are shown in Table 13.1.[8]

It should be noted at the outset that input–output structural decomposition studies generate, by definition, results at the sectoral level. For an n-sector model, each element in the n-element vector of changes – $\Delta\mathbf{x}$ in the case of gross outputs – will be decomposed into two or more constituent elements. This means that there is an inherent problem in finding appropriate summary measures of results in these studies. One obvious solution is to use total (economy-wide) figures – in the case of the decomposition in (13.7), this would be[9]

$$\mathbf{i}'(\Delta\mathbf{x}) = \underbrace{\mathbf{i}'[(1/2)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)]}_{\text{Economy-wide technology change effect}} + \underbrace{\mathbf{i}'[(1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})]}_{\text{Economy-wide final-demand change effect}}$$

Alternatives include grouping sectors into categories and then finding averages (simple or weighted) over the smaller numbers of elements in these groupings. For example: "fastest growing sectors" (say the top x percent), "slowest growing (fastest declining) sectors" (the bottom x percent) and other sectors [the middle $(100 - 2x)$ percent], or primary (natural resource related), secondary (manufacturing and processing) and tertiary (support and service oriented) sectors. As will be clear from this small example and from the empirical studies examined in section 13.2.5, any such economy-wide or

[7] There is some not very illuminating discussion in the literature about terms in (13.3) or (13.4) "absorbing" the interaction term. Starting with a rearranged (13.5), $[(\Delta\mathbf{L})\mathbf{f}^0 + (\Delta\mathbf{L})(\Delta\mathbf{f})] + \mathbf{L}^0(\Delta\mathbf{f}) \Rightarrow (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f})$, which is (13.4), and so $(\Delta\mathbf{L})\mathbf{f}^1$ incorporates the interaction term $[+(\Delta\mathbf{L})(\Delta\mathbf{f})]$. Equally plausible, however, is viewing (13.5) as $(\Delta\mathbf{L})\mathbf{f}^0 + [\mathbf{L}^0(\Delta\mathbf{f}) + (\Delta\mathbf{L})(\Delta\mathbf{f})] \Rightarrow (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f})$ which is (13.3), and now it is $\mathbf{L}^1(\Delta\mathbf{f})$ that has absorbed $[+(\Delta\mathbf{L})(\Delta\mathbf{f})]$. Similar rearrangements of (13.6) will show that $(\Delta\mathbf{L})\mathbf{f}^0$ in (13.3) or $\mathbf{L}^0(\Delta\mathbf{f})$ in (13.4) could be viewed as absorbing $[-(\Delta\mathbf{L})(\Delta\mathbf{f})]$. Mathematically, the result in (13.7) allocates one-half of the interaction term to technical change and one-half to final-demand change. See also Casler (2001) for thoughts on the interaction term.

[8] The reader can easily identify the various "absorptions" in the previous footnote in terms of the results in this table.

[9] Dividing both sides by n would generate one kind of "average" figure.

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Table 13.1 Alternative Structural Decompositions

	Technology Change Contribution	Final-Demand Change Contribution	Interaction Term
Equation (13.3)	$\begin{bmatrix} 0.90 \\ 8.62 \\ 9.01 \end{bmatrix}$	$\begin{bmatrix} 11.10 \\ 11.38 \\ 10.99 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Equation (13.4)	$\begin{bmatrix} 0.78 \\ 9.66 \\ 9.96 \end{bmatrix}$	$\begin{bmatrix} 11.22 \\ 10.34 \\ 10.04 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Equation (13.5)	$\begin{bmatrix} 0.90 \\ 8.62 \\ 9.01 \end{bmatrix}$	$\begin{bmatrix} 11.22 \\ 10.34 \\ 10.04 \end{bmatrix}$	$+\begin{bmatrix} -0.12 \\ 1.04 \\ 0.95 \end{bmatrix}$
Equation (13.6)	$\begin{bmatrix} 0.78 \\ 9.66 \\ 9.96 \end{bmatrix}$	$\begin{bmatrix} 11.10 \\ 11.38 \\ 10.99 \end{bmatrix}$	$-\begin{bmatrix} -0.12 \\ 1.04 \\ 0.95 \end{bmatrix}$
Equation (13.7)	$\begin{bmatrix} 0.84 \\ 9.14 \\ 9.49 \end{bmatrix}$	$\begin{bmatrix} 11.16 \\ 10.86 \\ 10.51 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Table 13.2 Sector-Specific and Economy-Wide Decomposition Results [Equation (13.7)]

	Output Change	Technology Change Contribution	Final-Demand Change Contribution
Sector 1	12	0.84 (7)	11.16 (93)
Sector 2	20	9.14 (46)	10.86 (54)
Sector 3	20	9.49 (47)	10.51 (53)
Economy-wide Total	52	19.47 (37)	32.53 (63)

averaging figures sweep an enormous amount of detail (and, usually, variation) under the rug.

Table 13.2 emphasizes the results from (13.7). Figures in parentheses indicate percentages of the total output change in each row. (Since these are hypothetical figures for illustration only, there is no need to be compulsive about detail in the percentages. We use no places to the right of the decimal.)

Of the economy-wide total output change in this example, 37 percent is seen to be attributable to technological change and 63 percent results from changes in final demand. But variation across sectors is large. The technology change contribution to individual sector output growth varies from 7 to 47 percent and (therefore) the final demand contribution varies from 53 to 93 percent.

13.1.2 Next-Level Decompositions: Digging Deeper into $\Delta\mathbf{f}$ and $\Delta\mathbf{L}$

Of course the story need not and does not end with the decompositions in (13.3)–(13.7). Changes in final demands, for example, may be the result of a change in the overall *level* of final demand or of a change in the relative proportions of expenditure on the various goods and services in the final-demand vector (the final-demand *mix*). Or, indeed, final-demand data may be collected and presented in several vectors, one for each final-demand *category*, such as household consumption, exports, government spending (federal, state, and local), and so on, and the relative importance of these categories may change.

Similarly, changes in the Leontief inverse result from changes in the economy's $\mathbf{A}$ matrix – which, in turn, may reflect various aspects of technology change, such as changes in production recipes (replacing metals with plastics in automobiles), substitutions caused by relative price changes (for domestically produced inputs and also for imports), reductions in a sector's materials inputs per unit of output brought about by economies of scale, and so on – as noted in section 7.2. We examine some approaches to account for these "next-level" effects. Before doing that, we need to generalize the decomposition results.

Additive Decompositions with Products of more than Two Terms The results in (13.7), above, can be looked at in the following way, which lends itself to generalization. Let $y^t = x_1^t x_2^t$ represent the general case in which the product of two variables (scalars, vectors, matrices or appropriate combinations) defines a dependent variable; the particular example here is $\mathbf{x}^t = \mathbf{L}^t\mathbf{f}^t$. Then the decompositions of $\Delta y = x_1^1 x_2^1 - x_1^0 x_2^0$ in (13.3) and (13.4) are seen to be of the form $\Delta y = (\Delta x_1)x_2^0 + x_1^1(\Delta x_2)$ and $\Delta y = (\Delta x_1)x_2^1 + x_1^0(\Delta x_2)$, respectively. Specifically, year-0 weights are to the right of a change term and year-1 weights are to the left in (13.3), and the year-0 and year-1 terms are reversed for (13.4).

An approach for the case of more than two terms, as in $y^t = x_1^t x_2^t \ldots x_n^t$, is to extend the logic of these two alternatives.[10] We begin with the case of $n = 3$, where $y^t = x_1^t x_2^t x_3^t$ and hence $\Delta y = x_1^1 x_2^1 x_3^1 - x_1^0 x_2^0 x_3^0$. Persistent and tedious substitutions from $x_1^1 = x_1^0 + \Delta x_1$, $x_2^1 = x_2^0 + \Delta x_2$ and $x_3^1 = x_3^0 + \Delta x_3$ will lead to

$$\Delta y = (\Delta x_1)x_2^0 x_3^0 + x_1^1(\Delta x_2)x_3^0 + x_1^1 x_2^1(\Delta x_3) \tag{13.8}$$

Alternative substitutions and rearrangements will generate

$$\Delta y = (\Delta x_1)x_2^1 x_3^1 + x_1^0(\Delta x_2)x_3^1 + x_1^0 x_2^0(\Delta x_3) \tag{13.9}$$

[10] These are not the only options. See Dietzenbacher and Los (1998) for a very thorough discussion of alternatives.

The usual averaging leads to

$$\Delta y = (1/2)(\Delta x_1)(x_2^0x_3^0 + x_2^1x_3^1) + (1/2)[x_1^0(\Delta x_2)x_3^1 + x_1^1(\Delta x_2)x_3^0] + (1/2)(x_1^0x_2^0 + x_1^1x_2^1)(\Delta x_3) \quad (13.10)$$

[Notice that the (1/2) terms result from averaging the *two* expressions for Δy in (13.8) and (13.9). They are unrelated to the *number* of elements in each of the terms on the right-hand sides of Δy.]

There are similar results for $n > 3$. The pattern is the same in the equations parallel to (13.8) and (13.9) – year-0 (year-1) weights always appear on the right of the Δx term and year-1 (year-0) weights always appear on the left. The generalization is straightforward but, again, the algebra is tedious. The parallel to (13.8) is

$$\Delta y = (\Delta x_1)(x_2^0 \ldots x_n^0) + x_1^1(\Delta x_2)(x_3^0 \ldots x_n^0) + \cdots + (x_1^1 \ldots x_{n-2}^1)(\Delta x_{n-1})x_n^0 + (x_1^1 \ldots x_{n-1}^1)(\Delta x_n) \quad (13.11)$$

The parallel to (13.9) has exactly the structure of (13.11) with superscripts "0" and "1" reversed. We write out the n-variable extension of (13.10), for completeness.

$$\begin{aligned}\Delta y = &(1/2)(\Delta x_1)[(x_2^0 \ldots x_n^0) + (x_2^1 \ldots x_n^1)] \\ &+ (1/2)[x_1^0(\Delta x_2)(x_3^1 \ldots x_n^1) + x_1^1(\Delta x_2)(x_3^0 \ldots x_n^0)] \\ &+ \cdots + (1/2)[(x_1^0 \ldots x_{n-2}^0)(\Delta x_{n-1})x_n^1 + (x_1^1 \ldots x_{n-2}^1)(\Delta x_{n-1})x_n^0] \\ &+ (1/2)[(x_1^0 \ldots x_{n-1}^0) + (x_1^1 \ldots x_{n-1}^1)](\Delta x_n) \quad (13.12)\end{aligned}$$

Changes in Final Demand Among the factors that may contribute to changes in final demands between two periods are: (1) the total amount of all expenditures for final demands – the final-demand *level*; (2) the *distribution* of total expenditure across final-demand categories – for example, the total value of household consumption, exports (possibly broken down by countries of destination), government expenditures (possibly separated into federal, state, and local), and other final demands, as proportions of total final-demand expenditure; and (3) the *product mix* within each particular final-demand category – for example, the proportion of total household consumption expenditure that goes to computers and computer services. This is reflected in the coefficients in the bridge matrix (see below).

In an n-sector input–output model, if there are p categories of final demand – instead of a single final-demand vector, $\underset{(n\times 1)}{\mathbf{f}^t}$ – then we have a final-demand matrix, $\underset{(n\times p)}{\mathbf{F}^t} = [\mathbf{f}_1^t, \ldots, \mathbf{f}_p^t]$, where $\mathbf{f}_k^t = \begin{bmatrix} f_{1k}^t \\ \vdots \\ f_{nk}^t \end{bmatrix}$, and f_{ik}^t records the amount of expenditure by final-demand category k on the product of sector i in year t. In particular,

a. $\mathbf{F}^t\mathbf{i} = \mathbf{f}^t$, the n-element vector of total final-demand deliveries from each sector in year t.
b. $\mathbf{i}'\mathbf{F}^t\mathbf{i} = \mathbf{i}'\mathbf{f}^t = f^t$, the *level* (total amount) of final-demand expenditure over all sectors in year t.
c. Let $\mathbf{y}^t = (\mathbf{i}'\mathbf{F}^t)' = \begin{bmatrix} y_1^t \\ \vdots \\ y_p^t \end{bmatrix}$, where y_k^t = total final-demand expenditure by final-demand category k in year t).

The vector that indicates the *distribution* of f^t across the p final-demand categories is found as the column sums of $\mathbf{F}^t$ divided by f^t, or

$$\underset{(p\times 1)}{\mathbf{d}^t} = [d_k^t] = (1/f^t)\mathbf{y}^t = \begin{bmatrix} y_1^t/f^t \\ \vdots \\ y_p^t/f^t \end{bmatrix} \qquad (13.13)$$

So d_k^t represents the *proportion* of total final-demand expenditure in year t that originated in category k. Finally, the bridge (*product mix*) matrix, $\underset{(n\times p)}{\mathbf{B}^t}$, is

$$\mathbf{B}^t = [b_{ik}^t] = (\mathbf{F}^t)(\hat{\mathbf{y}}^t)^{-1} \qquad (13.14)$$

So $\mathbf{B}^t$ is $\mathbf{F}^t$ normalized by its column sums – $b_{ik}^t = f_{ik}^t/y_k^t$ indicates the *proportion* of total expenditures by final-demand category k that was spent on the product of sector i in year t.[11]

With these definitions,

$$\mathbf{f}^t = f^t\mathbf{B}^t\mathbf{d}^t = \mathbf{B}^t\mathbf{y}^t \qquad (13.15)$$

and so

$$\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0 = f^1\mathbf{B}^1\mathbf{d}^1 - f^0\mathbf{B}^0\mathbf{d}^0 = \mathbf{B}^1\mathbf{y}^1 - \mathbf{B}^0\mathbf{y}^0 \qquad (13.16)$$

This holds for data with either only one final-demand vector ($p = 1$) or with several final-demand categories ($p > 1$). In the former case, $\mathbf{F}^t = \mathbf{f}^t = \begin{bmatrix} f_1^t \\ \vdots \\ f_n^t \end{bmatrix}$, $f^t = \mathbf{y}^t$ (a scalar), $\mathbf{B}^t$ is a column vector ($b_i^t = f_i^t/f^t = f_i^t/\mathbf{y}^t$) and $\mathbf{d}^t = 1$. In the latter case, the final-demand matrix, disaggregated by categories, is seen to be $\mathbf{F}^t = \mathbf{B}^t\hat{\mathbf{y}}^t$.

[11] This use of **B** is not to be confused with the output coefficients matrix in the Ghosh model.

Decomposing the final-demand change in (13.16) as in (13.8), (13.9), and (13.10) gives

$$\Delta\mathbf{f} = (\Delta f)\mathbf{B}^0\mathbf{d}^0 + f^1(\Delta\mathbf{B})\mathbf{d}^0 + f^1\mathbf{B}^1(\Delta\mathbf{d}) \qquad (13.17)$$

$$\Delta\mathbf{f} = (\Delta f)\mathbf{B}^1\mathbf{d}^1 + f^0(\Delta\mathbf{B})\mathbf{d}^1 + f^0\mathbf{B}^0(\Delta\mathbf{d}) \qquad (13.18)$$

and

$$\begin{aligned}\Delta\mathbf{f} = &\underbrace{(1/2)(\Delta f)(\mathbf{B}^0\mathbf{d}^0 + \mathbf{B}^1\mathbf{d}^1)}_{\text{Final-demand } level \text{ effect}} \\ &+ \underbrace{(1/2)[f^0(\Delta\mathbf{B})\mathbf{d}^1 + f^1(\Delta\mathbf{B})\mathbf{d}^0]}_{\text{Final-demand } mix \text{ effect}} \\ &+ \underbrace{(1/2)(f^0\mathbf{B}^0 + f^1\mathbf{B}^1)(\Delta\mathbf{d})}_{\text{Final-demand } distribution \text{ effect}}\end{aligned} \qquad (13.19)$$

When $p = 1$, $\Delta\mathbf{d} = 0$, and the third terms disappear from (13.17)–(13.19); in fact, (13.19) is simplified to

$$\Delta\mathbf{f} = \underbrace{(1/2)(\Delta f)(\mathbf{B}^0 + \mathbf{B}^1)}_{\text{Final-demand level effect}} + \underbrace{(1/2)(f^0 + f^1)(\Delta\mathbf{B})}_{\text{Final-demand mix effect}} \qquad (13.20)$$

13.1.3 Numerical Examples

One Category of Final Demand ($p = 1$) Continuing with the same numerical illustration,[12]

$$\mathbf{B}^0 = \begin{bmatrix} .45 \\ .3 \\ .25 \end{bmatrix}, \quad \mathbf{B}^1 = \begin{bmatrix} .4505 \\ .3153 \\ .2342 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} .0005 \\ .0153 \\ -.0158 \end{bmatrix}, \quad f^1 = 111, \quad f^0 = 100$$

Notice that (by definition) the column sums in $\mathbf{B}^0$ and $\mathbf{B}^1$ must be one and so the column sum in $\Delta\mathbf{B}$ must be zero; there must be one or more negative elements in $\Delta\mathbf{B}$ to balance one or more positive elements. This means that the final-demand mix effect for at least one sector – the second term in (13.20) – must be negative. In this numerical illustration, sector 3 has become relatively less important in total final-demand spending. Using (13.20) leads to the results shown in Table 13.3.

Two Categories of Final Demand ($p = 2$) Suppose that data are available on two categories of final demand – for example, households and all other final demand.

[12] It is necessary to work with more than two decimal places in these calculations, but results will continue to be rounded to two.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 601.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=635

Table 13.3 Sector-Specific and Economy-Wide Decomposition Results (with Two-Factor Final-Demand Decomposition Detail)[a]

	Output Change	Final-Demand Change Contribution		
		Level	Mix	Total
Sector 1	12	11.05 (92)	.11 (1)	11.16 (93)
Sector 2	20	9.35 (47)	1.51 (7)	10.86 (54)
Sector 3	20	11.45 (57)	−.94 (−5)	10.51 (53)
Total	52	31.85 (61)	.68 (1)	32.53 (63)

[a] In this and later tables, percentages are shown with no decimal places, so there may be (small) discrepancies between the total effect and the sum of its parts.

Consistent with the numerical illustration, let

$$\mathbf{F}^0 = [\mathbf{f}_1^0\ \mathbf{f}_2^0] = \begin{bmatrix} 20 & 25 \\ 10 & 20 \\ 15 & 10 \end{bmatrix} \quad \text{and} \quad \mathbf{F}^1 = [\mathbf{f}_1^1\ \mathbf{f}_2^1] = \begin{bmatrix} 25 & 25 \\ 15 & 20 \\ 18 & 8 \end{bmatrix}$$

Then

$$\mathbf{d}^0 = \begin{bmatrix} 45/100 \\ 55/100 \end{bmatrix} = \begin{bmatrix} 0.4500 \\ 0.5500 \end{bmatrix} \quad \text{and} \quad \mathbf{d}^1 = \begin{bmatrix} 58/111 \\ 53/111 \end{bmatrix} = \begin{bmatrix} 0.5225 \\ 0.4775 \end{bmatrix}$$

and the bridge matrices are

$$\mathbf{B}^0 = \begin{bmatrix} 20 & 25 \\ 10 & 20 \\ 15 & 10 \end{bmatrix} \begin{bmatrix} 1/45 & 0 \\ 0 & 1/55 \end{bmatrix} = \begin{bmatrix} 0.4444 & 0.4545 \\ 0.2222 & 0.3636 \\ 0.3333 & 0.1818 \end{bmatrix} \quad \text{and}$$

$$\mathbf{B}^1 = \begin{bmatrix} 25 & 25 \\ 15 & 20 \\ 18 & 8 \end{bmatrix} \begin{bmatrix} 1/58 & 0 \\ 0 & 1/53 \end{bmatrix} = \begin{bmatrix} 0.4310 & 0.4717 \\ 0.2586 & 0.3774 \\ 0.3103 & 0.1509 \end{bmatrix}$$

Finally,

$$\Delta\mathbf{d} = \begin{bmatrix} .0725 \\ -.0725 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} -.0134 & .0172 \\ .0364 & .0137 \\ -.0230 & -.0309 \end{bmatrix}, \quad \Delta f = 11$$

The decomposition in (13.19) generates the results in Table 13.4. Notice that, again by definition, column sums in $\Delta\mathbf{d}$ (as with $\Delta\mathbf{B}$) must be zero. This introduces negative elements into both the final-demand mix and distribution effects [the second and third terms in (13.19)].

13.1.4 Changes in the Direct Inputs Matrix

Decomposition of $\Delta\mathbf{L}$ Changes in the Leontief inverse between two time periods reflect, of course, changes in the underlying direct inputs matrices. One

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 602.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=636

Table 13.4 Sector-Specific and Economy-Wide Decomposition Results (with Three-Factor Final-Demand Decomposition Detail)

	Output Change	Final-Demand Change Contribution			
		Level	Mix	Distribution	Total
Sector 1	12	11.05 (92)	.31 (3)	−.21 (−2)	11.16 (93)
Sector 2	20	9.35 (47)	2.42 (12)	−.91 (−5)	10.86 (54)
Sector 3	20	11.45 (57)	−1.65 (−8)	.71 (4)	10.51 (53)
Total	52	31.85 (61)	1.08 (2)	−.41 (−1)	32.53 (63)

approach to translating $\Delta\mathbf{A}$ into $\Delta\mathbf{L}$ proceeds as follows. Given $\mathbf{L}^1 = (\mathbf{I} - \mathbf{A}^1)^{-1}$ and $\mathbf{L}^0 = (\mathbf{I} - \mathbf{A}^0)^{-1}$, postmultiply $\mathbf{L}^1$ through by $(\mathbf{I} - \mathbf{A}^1)$

$$\mathbf{L}^1(\mathbf{I} - \mathbf{A}^1) = \mathbf{I} = \mathbf{L}^1 - \mathbf{L}^1\mathbf{A}^1 \tag{13.21}$$

and premultiply $\mathbf{L}^0$ through by $(\mathbf{I} - \mathbf{A}^0)$

$$(\mathbf{I} - \mathbf{A}^0)\mathbf{L}^0 = \mathbf{I} = \mathbf{L}^0 - \mathbf{A}^0\mathbf{L}^0 \tag{13.22}$$

Rearrange (13.21) and postmultiply by $\mathbf{L}^0$

$$\mathbf{L}^1 - \mathbf{I} = \mathbf{L}^1\mathbf{A}^1 \Rightarrow \mathbf{L}^1\mathbf{L}^0 - \mathbf{L}^0 = \mathbf{L}^1\mathbf{A}^1\mathbf{L}^0 \tag{13.23}$$

Similarly, rearrange (13.22) and premultiply by $\mathbf{L}^1$

$$\mathbf{L}^0 - \mathbf{I} = \mathbf{A}^0\mathbf{L}^0 \Rightarrow \mathbf{L}^1\mathbf{L}^0 - \mathbf{L}^1 = \mathbf{L}^1\mathbf{A}^0\mathbf{L}^0 \tag{13.24}$$

Finally, subtract (13.24) from (13.23)

$$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \mathbf{L}^1\mathbf{A}^1\mathbf{L}^0 - \mathbf{L}^1\mathbf{A}^0\mathbf{L}^0 = \mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^0 \tag{13.25}$$

This expression relates the change in the Leontief inverse to the change in $\mathbf{A}$; the decomposition is a multiplicative one in which $\Delta\mathbf{A}$ is "doubly weighted" – in this case by $\mathbf{L}^1$ on the left and by $\mathbf{L}^0$ on the right. The reader can verify that changing each premultiplication to a postmultiplication, and vice versa, in deriving (13.21) through (13.24) will generate the (possibly surprising[13]) result that, in addition,

$$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \mathbf{L}^0\mathbf{A}^1\mathbf{L}^1 - \mathbf{L}^0\mathbf{A}^0\mathbf{L}^1 = \mathbf{L}^0(\Delta\mathbf{A})\mathbf{L}^1 \tag{13.26}$$

Since there is only one term on the right in either (13.25) or (13.26), there is no need to express $\Delta\mathbf{L}$ as the average of the two expressions; either one will do.

Again, interaction terms will appear if we choose to have only year-0 ($\mathbf{L}^0$) or only year-1 ($\mathbf{L}^1$) weights. For example, replacing $\mathbf{L}^1$ with $\mathbf{L}^0 + \Delta\mathbf{L}$ in (13.25) leads to

[13] The result is surprising in the sense that the order in which matrices appear in matrix multiplication usually makes a difference in the outcome (in contrast to scalar multiplication).

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 603.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=637

$\Delta\mathbf{L} = \mathbf{L}^0(\Delta\mathbf{A})\mathbf{L}^0 + (\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^0$. Making the same replacement in (13.26) generates $\Delta\mathbf{L} = \mathbf{L}^0(\Delta\mathbf{A})\mathbf{L}^0 + \mathbf{L}^0(\Delta\mathbf{A})(\Delta\mathbf{L})$. This identifies another instance in which the general "order makes a difference" rule in matrix algebra is violated; since the second terms must be equal, we see that $(\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^0 = \mathbf{L}^0(\Delta\mathbf{A})(\Delta\mathbf{L})$. Also, substituting $\mathbf{L}^1 - \Delta\mathbf{L}$ for $\mathbf{L}^0$ in both (13.25) and (13.26) will produce $\Delta\mathbf{L} = \mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^1 - \mathbf{L}^1(\Delta\mathbf{A})(\Delta\mathbf{L})$ and $\Delta\mathbf{L} = \mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^1 - (\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^1$, respectively. In these two cases, we also find that the interaction terms are equal – $(\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^1 = \mathbf{L}^1(\Delta\mathbf{A})(\Delta\mathbf{L})$.

In what follows, we will use the result in (13.25) to convert changes in the Leontief inverse into changes in the **A** matrix.[14]

Decomposition of $\Delta\mathbf{A}$ There are many ways to create decompositions of $\Delta\mathbf{A}$. For example, the RAS procedure has been proposed as a descriptive device to identify the underlying causes of coefficient change between $\mathbf{A}_0$ and $\mathbf{A}_1$ when both matrices are known. This is entirely different from the usual use of RAS, which is to estimate an unknown $\mathbf{A}_1$ when only $\mathbf{u}_1$, $\mathbf{v}_1$, and $\mathbf{x}_1$ are known.[15] [A general introduction to RAS is given in Chapter 7; there we used $\mathbf{A}(0)$, $\mathbf{u}(1)$ and so on rather than $\mathbf{A}_0$, $\mathbf{u}_1$, etc.] From $\tilde{\mathbf{A}}_1 = \hat{\mathbf{r}}\mathbf{A}_0\hat{\mathbf{s}}$ and when $\tilde{\mathbf{A}}_1 \neq \mathbf{A}_1$ (as is generally the case), let $\mathbf{D} = \mathbf{A}_1 - \tilde{\mathbf{A}}_1 = \mathbf{A}_1 - \hat{\mathbf{r}}\mathbf{A}_0\hat{\mathbf{s}}$ or $d_{ij} = a_{ij}^1 - \tilde{a}_{ij}^1 = a_{ij}^1 - r_i a_{ij}^0 s_j$; so $\mathbf{A}_1 = \tilde{\mathbf{A}}_1 + \mathbf{D} = \hat{\mathbf{r}}\mathbf{A}_0\hat{\mathbf{s}} + \mathbf{D}$ and $a_{ij}^1 = r_i a_{ij}^0 s_j + d_{ij}$. This allows the separation of coefficient changes into those that are column-specific (fabrication effects for sector j, captured in s_j), row-specific (substitution effects in sector i, reflected in r_i) and cell-specific (that part of the change in a_{ij} that is caused by other circumstances, d_{ij}).[16]

Here we illustrate a straightforward disaggregation into column-specific changes only. Since each column in **A** reflects a sector's production recipe, identifying the changes column by column is one way of disentangling the effects of input changes in each of the sectors in the economy. For expositional simplicity, we denote these as technology change. (The interested reader might refer again to section 7.2 where alternative and more aggregate approaches to measuring changes in coefficients were explored.)

For an n-sector economy,

$$\mathbf{A}^1 = \mathbf{A}^0 + \Delta\mathbf{A} = \begin{bmatrix} a_{11}^0 + \Delta a_{11} & \cdots & a_{1n}^0 + \Delta a_{1n} \\ \vdots & & \vdots \\ a_{n1}^0 + \Delta a_{n1} & \cdots & a_{nn}^0 + \Delta a_{nn} \end{bmatrix}$$

[14] A continuous version of this approach has been noted (for example, Afrasiabi and Casler, 1991, Rose and Casler, 1996). As in (13.21), with $\mathbf{L}(\mathbf{I} - \mathbf{A}) = \mathbf{L} - \mathbf{L}\mathbf{A} = \mathbf{I}$, use the product rule for differentiation, $(d\mathbf{L}/dt) - (d\mathbf{L}/dt)\mathbf{A} - \mathbf{L}(d\mathbf{A}/dt) = 0$ or $(d\mathbf{L}/dt)(\mathbf{I} - \mathbf{A}) = \mathbf{L}(d\mathbf{A}/dt)$. Postmultiplying by $\mathbf{L}$, $(d\mathbf{L}/dt) = \mathbf{L}(d\mathbf{A}/dt)\mathbf{L}$.

[15] For example, see van der Linden and Dietzenbacher, 2000, Dietzenbacher and Hoekstra, 2002; also de Mesnard, 2004, 2006.

[16] As argued in van der Linden and Dietzenbacher (2000, pp. 2208–2209), a poor RAS performance simply indicates that other [cell specific] determinants need to be taken into account. These provide the necessary corrections whenever the fabrication effects and substitution effects alone do not adequately capture the coefficient changes.

Let $\Delta\mathbf{A}^{(j)} = \begin{bmatrix} 0 & \cdots & \Delta a_{1j} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \Delta a_{nj} & \cdots & 0 \end{bmatrix}$ represent changes in sector j's technology – the superscript "(j)" identifies the sector (column) in which coefficients change.[17] Then

$$\Delta\mathbf{A} = \Delta\mathbf{A}^{(1)} + \cdots + \Delta\mathbf{A}^{(j)} + \cdots + \Delta\mathbf{A}^{(n)} = \sum_{j=1}^{n} \underbrace{\Delta\mathbf{A}^{(j)}}_{\substack{\text{Technology change} \\ \text{in sector } j}} \tag{13.27}$$

This decomposition of $\Delta\mathbf{A}$ can be introduced into (13.25), and the resulting expression for $\Delta\mathbf{L}$ can then be used in (13.7), which now looks like this:

$$\begin{aligned}
\Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\
&= [(1/2)\mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\
&= [(1/2)\mathbf{L}^1(\Delta\mathbf{A}^{(1)} + \cdots + \Delta\mathbf{A}^{(n)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\
&= \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(1)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 1}} + \cdots + \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(n)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector } n} \\
&\quad + \underbrace{(1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Effect of final-demand change}}
\end{aligned} \tag{13.28}$$

Numerical Illustration (continued) For our numerical example,

$$\mathbf{A}^0 = \begin{bmatrix} .1000 & .2500 & .2500 \\ .1500 & .0625 & .3000 \\ .3000 & .5000 & .0500 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^1 = \begin{bmatrix} .1071 & .1500 & .2917 \\ .2143 & .1100 & .2500 \\ .3214 & .5000 & .0667 \end{bmatrix}$$

so

$$\Delta\mathbf{A} = \begin{bmatrix} .0071 & -.1 & .0417 \\ .0643 & .0475 & -.0500 \\ .0214 & 0 & .0167 \end{bmatrix}$$

and, in particular,

$$\Delta\mathbf{A}^{(1)} = \begin{bmatrix} .0071 & 0 & 0 \\ .0643 & 0 & 0 \\ .0214 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{(2)} = \begin{bmatrix} 0 & -.1 & 0 \\ 0 & .0475 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{(3)} = \begin{bmatrix} 0 & 0 & .0417 \\ 0 & 0 & -.0500 \\ 0 & 0 & .0167 \end{bmatrix}$$

[17] The superscript parentheses serve to distinguish $\mathbf{A}^1$, the direct inputs matrix in period 1, from $\Delta\mathbf{A}^{(1)}$, the matrix that reflects the technology *change* in sector 1 only.

Table 13.5 Sector-Specific and Economy-Wide Decomposition Results (with Additional Technology and Final-Demand Decomposition Detail)

		Technology Change Contribution				Final-Demand Change Contribution			
	Output Change	Sector 1	Sector 2	Sector 3	Total	Level	Mix	Distri-bution	Total
Sector 1	12	6.64 (55)	−10.25 (−85)	4.45 (37)	.84 (7)	11.05 (92)	.31 (3)	−.21 (−2)	11.16 (93)
Sector 2	20	12.42 (62)	1.28 (6)	−4.56 (−23)	9.14 (46)	9.35 (47)	2.42 (12)	−.91 (−5)	10.86 (54)
Sector 3	20	11.37 (57)	−2.85 (−14)	.97 (5)	9.49 (47)	11.45 (57)	−1.65 (−8)	.71 (4)	10.51 (53)
Total	52	30.43 (59)	−11.82 (−23)	.86 (2)	19.47 (37)	31.85 (61)	1.08 (2)	−.41 (−1)	32.53 (63)

Table 13.5 indicates the additional results from using this technology change decomposition for our numerical illustration. As usual, percentages of row totals are in parentheses. (Final-demand results repeat those in Table 13.4.)

13.1.5 Decompositions of Changes in Some Function of x

A number of studies have decomposed not simply gross output change but rather changes in some variable that depends on output. For example, if we have a set of labor input coefficients – employment per dollar of output in sector j at time t (e_j^t) – let $(\mathbf{e}^t)' = [e_1^t, \ldots, e_n^t]$. Then the vector of employment, by sector, associated with output at t will be $\boldsymbol{\varepsilon}^t = \hat{\mathbf{e}}^t\mathbf{x}^t = \hat{\mathbf{e}}^t\mathbf{L}^t\mathbf{f}^t$, and the vector of changes in employment is

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^0 = \hat{\mathbf{e}}^1\mathbf{L}^1\mathbf{f}^1 - \hat{\mathbf{e}}^0\mathbf{L}^0\mathbf{f}^0 \tag{13.29}$$

Decomposition into contributions by the three elements now follows the standard pattern shown in (13.10). Here this means

$$\Delta\boldsymbol{\varepsilon} = (1/2)\underbrace{(\Delta\hat{\mathbf{e}})(\mathbf{L}^0\mathbf{f}^0 + \mathbf{L}^1\mathbf{f}^1)}_{\text{Labor input coefficient change}} + (1/2)\underbrace{[\hat{\mathbf{e}}^0(\Delta\mathbf{L})\mathbf{f}^1 + \hat{\mathbf{e}}^1(\Delta\mathbf{L})\mathbf{f}^0]}_{\text{Technology change}} + (1/2)\underbrace{(\hat{\mathbf{e}}^0\mathbf{L}^0 + \hat{\mathbf{e}}^1\mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change}} \tag{13.30}$$

Of course, additional decompositions of $\Delta\mathbf{L}$ and/or $\Delta\mathbf{f}$ as in section 13.1.2 are possible. Exactly the same principles apply for any economic variable that is related to output by a similar set of coefficients per dollar of sectoral output – pollution generation, energy consumption, value added, etc.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 606.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=640

13.1.6 Summary for $\Delta\mathbf{x}$

For $\Delta\mathbf{x}$ we assemble both the final-demand decomposition (including distribution across final-demand categories) and the technology change decomposition in the same expression, primarily for completeness. The expression includes all six of the change components.

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{f}^0+\mathbf{f}^1) + (1/2)(\mathbf{L}^0+\mathbf{L}^1)(\Delta\mathbf{f})\\ &= \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(1)})\mathbf{L}^0](\mathbf{f}^0+\mathbf{f}^1)}_{\text{Effect of technology change in sector 1}} + \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(2)})\mathbf{L}^0](\mathbf{f}^0+\mathbf{f}^1)}_{\text{Effect of technology change in sector 2}}\\ &\quad + \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(3)})\mathbf{L}^0](\mathbf{f}^0+\mathbf{f}^1)}_{\text{Effect of technology change in sector 3}} + \underbrace{(1/4)(\mathbf{L}^0+\mathbf{L}^1)(\Delta f)(\mathbf{P}^0\mathbf{d}^0+\mathbf{P}^1\mathbf{d}^1)}_{\text{Effect of change in final-demand level}}\\ &\quad + \underbrace{(1/4)(\mathbf{L}^0+\mathbf{L}^1)[f^0(\Delta\mathbf{P})\mathbf{d}^1+f^1(\Delta\mathbf{P})\mathbf{d}^0]}_{\text{Effect of change in final-demand mix}} + \underbrace{(1/4)(\mathrm{f}^0\mathbf{P}^0+\mathrm{f}^1\mathbf{P}^1)(\Delta\mathbf{d})}_{\text{Effect of change in final-demand distribution}}\end{aligned} \tag{13.31}$$

13.1.7 SDA in a Multiregional Input–Output (MRIO) Model

The standard form of the MRIO model (Chapter 3) is $\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf} = \tilde{\mathbf{L}}\mathbf{Cf}$, where $\tilde{\mathbf{L}} = (\mathbf{I} - \mathbf{CA})^{-1}$, $\mathbf{A}$ is a technical coefficients matrix indicating intermediate inputs for each region from both within and outside of the region and $\mathbf{C}$ contains input proportions (both intraregional and interregional shipments). The distinctive feature of this formulation is that the Leontief-like inverse contains both technical coefficients and trade proportions.

Following (13.10), for $\mathbf{x} = \tilde{\mathbf{L}}\mathbf{Cf}$ we have

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)(\Delta\tilde{\mathbf{L}})(\mathbf{C}^0\mathbf{f}^0 + \mathbf{C}^1\mathbf{f}^1) + (1/2)[\tilde{\mathbf{L}}^0(\Delta\mathbf{C})\mathbf{f}^1 + \tilde{\mathbf{L}}^1(\Delta\mathbf{C})\mathbf{f}^0]\\ &\quad + (1/2)(\tilde{\mathbf{L}}^0\mathbf{C}^0 + \tilde{\mathbf{L}}^1\mathbf{C}^1)(\Delta\mathbf{f})\end{aligned} \tag{13.32}$$

To disentangle the trade proportions and technical coefficients in $\tilde{\mathbf{L}}$ we follow (13.25) and then (13.7), namely

$$\Delta\tilde{\mathbf{L}} = \tilde{\mathbf{L}}^1(\Delta\mathbf{CA})\tilde{\mathbf{L}}^0$$

and

$$\Delta\mathbf{CA} = (1/2)(\Delta\mathbf{C})(\mathbf{A}^0 + \mathbf{A}^1) + (1/2)(\mathbf{C}^0 + \mathbf{C}^1)(\Delta\mathbf{A}) \tag{13.33}$$

First, using $\Delta\tilde{\mathbf{L}} = \tilde{\mathbf{L}}^1(\Delta\mathbf{CA})\tilde{\mathbf{L}}^0$ in (13.32),

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)[\tilde{\mathbf{L}}^1(\Delta\mathbf{CA})\tilde{\mathbf{L}}^0](\mathbf{C}^0\mathbf{f}^0 + \mathbf{C}^1\mathbf{f}^1) + (1/2)[\tilde{\mathbf{L}}^0(\Delta\mathbf{C})\mathbf{f}^1 + \tilde{\mathbf{L}}^1(\Delta\mathbf{C})\mathbf{f}^0]\\ &\quad + (1/2)(\tilde{\mathbf{L}}^0\mathbf{C}^0 + \tilde{\mathbf{L}}^1\mathbf{C}^1)(\Delta\mathbf{f})\end{aligned}$$

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 607.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=641

and then using (13.33) (and rearranging)

$$\Delta\mathbf{x} = \underbrace{(1/4)[\tilde{\mathbf{L}}^1(\mathbf{C}^0+\mathbf{C}^1)(\Delta\mathbf{A})\tilde{\mathbf{L}}^0](\mathbf{C}^0\mathbf{f}^0+\mathbf{C}^1\mathbf{f}^1)}_{\text{Effect of technology change}}$$
$$+\underbrace{(1/4)[\tilde{\mathbf{L}}^1(\Delta\mathbf{C})(\mathbf{A}^0+\mathbf{A}^1)\tilde{\mathbf{L}}^0](\mathbf{C}^0\mathbf{f}^0+\mathbf{C}^1\mathbf{f}^1)}_{\text{One effect of trade coefficient change}}$$
$$+\underbrace{(1/2)[\tilde{\mathbf{L}}^0(\Delta\mathbf{C})\mathbf{f}^1+\tilde{\mathbf{L}}^1(\Delta\mathbf{C})\mathbf{f}^0]}_{\text{A second effect of trade coefficient change}}+\underbrace{(1/2)(\tilde{\mathbf{L}}^0\mathbf{C}^0+\tilde{\mathbf{L}}^1\mathbf{C}^1)(\Delta\mathbf{f})}_{\text{Effect of final-demand change}} \quad (13.34)$$

Notice in particular that the change in trade proportions exerts influence in conjunction with both the technical coefficients ($\mathbf{A}^0$ and $\mathbf{A}^1$) and also the final demands ($\mathbf{f}^0$ and $\mathbf{f}^1$). This is logical, since in the MRIO model both $\mathbf{A}$ and $\mathbf{f}$ are transformed – into $\mathbf{CA}$ and $\mathbf{Cf}$.

Embellishments are possible. For example, the final-demand effect might be further decomposed into level, mix and/or distribution, as in section 13.1.2. Furthermore, some models may feature (or at least propose) separate trade proportions for intermediate inputs and for final demands, leading to $\mathbf{x} = (\mathbf{I} - \mathbf{C}_a\mathbf{A})^{-1}\mathbf{C}_f\mathbf{f} = \tilde{\mathbf{L}}^*\mathbf{C}_f\mathbf{f}$. In that case, $\Delta\mathbf{C}_a$ and $\Delta\mathbf{C}_f$ must be treated separately. This simply leads to more complexity (more terms) in (13.34). In Appendix 13.1 we explore the implications of alternative groupings of the terms in $\mathbf{x} = \tilde{\mathbf{L}}\mathbf{C}\mathbf{f}$ (as has been done in some published studies) into either $\mathbf{x} = \mathbf{M}\mathbf{f}$, where $\mathbf{M} = \tilde{\mathbf{L}}\mathbf{C}$, or $\mathbf{x} = \tilde{\mathbf{L}}\mathbf{y}$, where $\mathbf{y} = \mathbf{C}\mathbf{f}$.

13.1.8 Empirical Examples

Analysts are generally interested in structural decompositions because they offer a means of quantifying the relative importance of various components in an "explanation" of some observed economic change – in early studies this was usually changes in industry outputs; more recently, changes in labor use, value added, energy use, pollution emissions, service industry outputs, etc. have also been decomposed. The results of empirical SDA studies are often used to inform policy decisions – the relative importance of trade (and hence trade policy) to an economy, the relative importance of one or more components of final demand (and hence tax or subsidy policy), and so forth. As noted earlier, decompositions generate results at a sectoral level and summary measures are needed. In Tables 13.6, 13.7, and 13.9, below, virtually all of the rich detail in each of the studies cited has been foregone in favor of simple averages in order to present figures that are comparable across studies.

Studies Using National Models The first study known to us that uses this approach is Chenery, Shishido and Watanabe (1962), for Japan over the periods 1914–1935 and 1935–1954.[18] The authors were interested in deviations of later year output

[18] This builds on earlier work by Chenery (for example, Chenery, 1960). A thorough summary of this kind of analysis in the economic development literature can be found in Syrquin (1988). Illustrative examples include

from what it would have been under a regime of proportional growth from an earlier year. These deviations were decomposed into the effects of (1) changes in domestic final demand, (2) changes in exports, (3) changes in imports and (4) changes in technology (as represented by changes in elements of the $\mathbf{A}$ matrix).

A study by Vaccara and Simon (1968), to the best of our knowledge, represents the first application of this kind of decomposition approach to the US economy. Using 42 industry groups, they measured the amount of output change that was attributable to final-demand change and the amount due to coefficient change over the 1947–1958 period. As a (very) general conclusion, they found final-demand changes somewhat more important than changes in technical coefficients in contributing to overall output change over the period.

Bezdek and Wendling (1976) continued this kind of analysis. They factored $\Delta\mathbf{x}$ into final-demand and coefficient change in a 75-sector model of the US economy for the 1947–1958, 1958–1963, and 1963–1966 periods. In addition, they compared their decomposition results for 1958–1963 with those reported for Germany (1958–1962) in Stäglin and Wessels (1972) at a 35-sector level. They found similarity in the industry-specific influences of final-demand change but not of coefficient change.

The late 1980s and early 1990s saw the beginnings of an explosion of empirical studies using SDA. The work of Feldman, McClain and Palmer (1987) is frequently cited.[19] This study also examined the relative importance in the US economy of changes in final demands and changes in technology – this time over the 1963–1978 period using a very disaggregated 400-sector level of analysis. (The 1978 table was an updated version of the 1972 survey-based national table.)

They use the form $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f} \Rightarrow \mathbf{x} = \mathbf{L}\mathbf{B}\mathbf{f}$ and then define $\mathbf{C} = \mathbf{L}\mathbf{B}$ so that $\mathbf{x} = \mathbf{C}\mathbf{f}$, where $\underset{(n\times p)}{\mathbf{B}}$ is the bridge matrix that connects the outputs of some $n =$ 400 sectors to $p = 160$ categories of final demand.[20] Thus their decomposition takes the form $\Delta\mathbf{x} = (\Delta\mathbf{C})\mathbf{f}^0 + \mathbf{C}^1(\Delta\mathbf{f})$ or $\Delta\mathbf{x} = (\Delta\mathbf{C})\mathbf{f}^1 + \mathbf{C}^0(\Delta\mathbf{f})$ – as in (13.3) and (13.4). They define structural change broadly – "including changes in the structure of production (technical change, reflected in changes in $\mathbf{A}$) and in the microstructure of expenditure (reflected in changes in $\mathbf{B}$)" (p. 504).[21] Generally speaking, the contribution made by coefficient change was larger than the contribution made by final-demand change for many of the fastest growing (termed "emerging") and fastest declining industries. At the same time, for most industries (almost 80 percent), the coefficient change component accounted (in absolute terms) for less than half of the gross output change.[22]

Fujita and James (1990) – and many other publications by these authors – at the national level and Siegel, Alwang and Johnson (1995) for a "growth accounting" study at the regional level.

19 And, less frequently, Feldman and Palmer (1985).

20 This use of $\mathbf{C}$ is not to be confused with the trade proportions matrix of the MRIO model.

21 They recognize that an alternative would be to group $\mathbf{B}$ with $\mathbf{f}$ and to use $\mathbf{x} = \mathbf{L}(\mathbf{B}\mathbf{f})$, leading to $\Delta\mathbf{x} = [\Delta\mathbf{L}](\mathbf{B}^0\mathbf{f}^0) + \mathbf{L}^1(\Delta\mathbf{B}\mathbf{f})$ and $\Delta\mathbf{x} = [\Delta\mathbf{L}](\mathbf{B}^1\mathbf{f}^1) + \mathbf{L}^0(\Delta\mathbf{B}\mathbf{f})$. See comments on the effect of alternative groupings on decompositions in Appendix 13.1.

22 Wolff (1985) used the same mode of analysis to study trends in productivity in the US economy.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 609.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=643

A second frequently cited study from this period is that by Skolka (1989). It describes the structural decomposition methodology in some detail and applies it to a 19-sector data set for Austria (1964–1976). Both net output (value added) change and employment change were decomposed into an intermediate demand (technology) component (with separate domestic and imports parts) and a final-demand component (with separate domestic and exports parts).

In what follows, we present brief overviews of several (from among many) additional empirical SDA studies concerned with identifying components of total output change (in chronological order). The main characteristics of these (and other studies) are summarized in Table 13.6.

1. Fujimagari (1989). Fujimagari suggests that bundling **L** and **B** together (as in Feldman, McClain and Palmer) is inappropriate. Instead he uses two tripartite decompositions and averages their results. These are

$$\Delta \mathbf{x} = (\Delta \mathbf{L})\mathbf{B}^0\mathbf{f}^0 + \mathbf{L}^1(\Delta \mathbf{B})\mathbf{f}^0 + \mathbf{L}^1\mathbf{B}^1(\Delta \mathbf{f}) \text{ and}$$
$$\Delta \mathbf{x} = (\Delta \mathbf{L})\mathbf{B}^1\mathbf{f}^1 + \mathbf{L}^0(\Delta \mathbf{B})\mathbf{f}^1 + \mathbf{L}^0\mathbf{B}^0(\Delta \mathbf{f})$$

[as in (13.8) and (13.9)] in a 189-sector Canadian model for 1961–1971 and 1971–1981. This approach has been used by others in later studies.

2. Barker (1990). Changes over 1979–1984 in the output of market service industries in the UK are investigated – including distribution, transport, communications, business services, and others. The decomposition – into changes internal to the services group, external to the group in the rest of manufacturing and external in the rest of industry – uses partitioned matrices extensively. Each of these is further decomposed into changes in: input–output coefficients, total final demand (level) and the structure of final demand (the distribution, as reflected in the bridge matrix).

3. Martin and Holland (1992). Changes over 1972–1977 in the output of some 477 US industries are decomposed from the defining equation

$$\mathbf{x}^t = (\mathbf{I} - \hat{\mathbf{u}}^t\mathbf{A}^t)^{-1}(\hat{\mathbf{u}}^t\mathbf{f}^t + \mathbf{e}^t) = \mathbf{L}^t(\hat{\mathbf{u}}^t\mathbf{f}^t + \mathbf{e}^t)$$

in which (all for year t) $\hat{\mathbf{u}}$ is a diagonal matrix containing the domestic supply ratio for each sector, $\mathbf{A}$ is the technical coefficient matrix (including imports), $\mathbf{f}$ is the domestic final-demand vector and $\mathbf{e}$ is a vector of exports. Thus $\hat{\mathbf{u}}^t\mathbf{A}^t$ is an estimate of the domestic direct input coefficients matrix and $\hat{\mathbf{u}}^t\mathbf{f}^t$ is an estimate of the vector of domestic final demand that is satisfied from domestic sources. The decomposition used is essentially that in (13.9), namely

$$\Delta \mathbf{x} = (\Delta \mathbf{L})(\hat{\mathbf{u}}^1\mathbf{f}^1 + \mathbf{e}^1) + \mathbf{L}^0(\Delta \mathbf{u})\mathbf{f}^1 + \mathbf{L}^0\hat{\mathbf{u}}^0(\Delta \mathbf{f}) + \mathbf{L}^0(\Delta \mathbf{e})$$

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 610.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=644

After a good deal of algebra this can be expressed as

$$\Delta \mathbf{x} = \mathbf{L}^0 \hat{\mathbf{u}}^0 (\Delta \mathbf{f}) + \mathbf{L}^0 (\Delta \mathbf{e}) + \mathbf{L}^0 (\Delta \mathbf{u})(\mathbf{f}^1 + \mathbf{A}^1 \mathbf{x}^1) + \mathbf{L}^0 \hat{\mathbf{u}}^0 (\Delta \mathbf{A}) \mathbf{x}^1$$

(No alternative decompositions were used and so the results were not averages.) The decomposition is thus apportioned to changes due to: domestic final demand, export demand, import substitution, and input–output coefficients. With results for 477 sectors, groupings were necessary – these included aggregations into: (1) primary (25 natural resource related industries), secondary (409 manufacturing and processing industries) and tertiary (43 support and service oriented industries); (2) nine sectors that represent the BEA one-digit aggregation level; and (3) the 30 fastest growing and the 30 slowest growing industries.

When commodity sectors were categorized according to 1972–1977 growth rates, the importance of the technical change contribution was seen to increase with categories of increasing growth or decline – results consistent with those in Feldman, McClain and Palmer (1987). At the same time, examination of the specific decompositions for the 30 fastest growing and 30 fastest declining sectors indicated that final demand was the dominant component in output change in 60 and 67 percent of the cases, respectively, whereas technical coefficient change was dominant in about 30 percent of the cases (both for rapidly growing and rapidly declining sectors). This view of their results is at variance with those of Feldman, McClain and Palmer.

4. Liu and Saal (2001). This study examines changes in gross outputs in South Africa over 1975–1993. It employs essentially the same decomposition as Martin and Holland (1992), except that final demand is separated into changes in private consumption, investment spending, government spending, exports, and import substitution.
5. Dietzenbacher and Hoekstra (2002). This study focuses on output change for 25 sectors in the Netherlands over 1975–1985. The Netherlands data are embedded in an intercountry model for the European Union, and final-demand categories include separate columns for exports to each of five EU member countries (Germany, France, Italy, Belgium, and Denmark), the rest of the EU, the rest of the world, household consumption, and other final demand. As might be expected, large differences were observed across sectors, countries, and final-demand categories.
6. Roy, Das and Chakraborty (2002). The particular interest of this study is to identify sources of growth in the information sectors in a 31-sector input–output model of the Indian economy over 1983–1984 to 1989–1990. Instead of partitioning the matrices into quadrants of information and non-information sectors (as in some of the energy studies noted below), the authors simply define a matrix $\hat{\mathbf{z}}$, created from an identity matrix by replacing the main-diagonal ones with zeros for all non-information sectors (so the remaining on-diagonal elements – and all off-diagonal elements – are 0). Then $\hat{\mathbf{z}}\mathbf{x}$ selects only the information rows from the results of various decompositions.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 611.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=645

Table 13.6 Selected Empirical Structural Decomposition Studies

Author(s) and Source	Details (country; dates; changed variable(s); aggregation level)	Decomposition Components (percentage of total change[a]) Technology	Final Demand
Feldman McClain and Palmer (1987, Table 1)	US; 1963–1978; $\Delta \mathbf{x}$; 400 sectors (results for 15 fastest growing industries)	62	38
Skolka (1989, pp. 59–60)	Austria; 1964–76; Δ (value added) and also Δ (employment); 19 industries	26 (v.a.), 34 (emp.)	74 (v.a.), 66 (emp.) *Domestic*: 18 (v.a.), 46 (emp.) *Foreign*: 56 (v.a.), 20 (emp.)
Fujimagari (1989, Tables 1 and 2)	Canada; 1961–71 and 1971–81; $\Delta \mathbf{x}$; 189 industries (results for 15 fastest and 15 slowest growing industries)	1961–71 28 (top 15), –86 (bottom 15) 1971–81 22 (top 15), 159 (bottom 15)	1961–71 72 (top 15) 186 (bottom 15) 1971–81 78 (top 15) –59 (bottom 15) *Level* 1961–71 38 (top 15) 69 (bottom 15) 1971–81 61 (top 15) –120 (bottom 15) *Mix* 1961–71 34 (top 15) 117 (bottom 15) 1971–81 17 (top 15) 61 (bottom 15)
Barker (1990, Table 4)	UK; 1979–84; $\Delta \mathbf{x}$ (service industries); 101 ind., 13 serv. ind. (aggregated to 5 serv. ind.)	63	18 *Level*: –1 *Mix*: 20
Martin and Holland (1992, Table 1)	US; 1972–77; $\Delta \mathbf{x}$; 477 sectors	6	94 *Domestic*: 81 *Export*: 23 *Import Use*: –10
Liu and Saal (2001, Table 5)	South Africa; 1975–93; $\Delta \mathbf{x}$; 34 and 10 sectors; results for 10 sectors only	28	72 (Pvt. Cons., 61; Gov. Cons., 7; Inv., –32; Exp. 29, Imp. Subs., 7)

Table 13.6 (*cont.*)

Author(s) and Source	Details (country; dates; changed variable(s); aggregation level)	Decomposition Components (percentage of total change[a]): Technology		Final Demand		
Dietzenbacher and Hoekstra (2002, Table 10.2)	The Netherlands; 1975–85; $\Delta\mathbf{x}$; 25 sectors	21[b] (−201, 135)		79[b] (−35, 301)		
				Level 78 (−118, 2458)	*Category* 1 (−2594, 257)	*Product Mix* 0 (−39, 236)
Roy, Das and Chakraborty (2002, Table 4)	India; 1983/4–89/90; $\Delta\mathbf{x}$ (information sectors); 30 non-information sectors plus 5 information sectors	3		97		
		Info. coeffs. 3	*Non-info. coeffs.* 0	*Domestic* 91 (Level 65, Mix 26)	*Exports* 6	*Import Substitution* 0

[a]Figures may not add to 100 percent due to rounding.
[b]Figures in parentheses indicate boundaries in the range of values across the 25 sectors in the study.

There have been many SDA studies concerned with energy and environmental issues; some are noted in Table 13.7.[23] Brief overviews of some of these are given below.

1. Office of Technology Assessment (US Congress, OTA, 1990). The primary interest of this study is to investigate the components of the change in energy use in the USA between 1972 and 1985. Final-demand level and mix along with changes in technology, disaggregated into energy inputs and non-energy inputs, are investigated. The decompositions are carried out for five energy types: coal, crude oil and gas, refined petroleum, primary electricity, and utility gas.

   The calculation of the change in energy use due to different economic factors was achieved by using 1985 as a base year and systematically varying one factor over time while holding all other factors constant in their 1985 form. The model separated energy sectors and other sectors and uses hybrid-units form (Chapter 9). The first k sectors (here $k = 5$) are energy commodities and energy industries. In partitioned form, the units in the four quadrants of the model are

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11}(\text{BTU/BTU}) & \mathbf{A}_{12}(\text{BTU/\$}) \\ \mathbf{A}_{21}(\text{\$/BTU}) & \mathbf{A}_{22}(\text{\$/\$}) \end{bmatrix},$$

[23] Early energy-use decomposition studies can be found in Casler and Hannon (1989), or Casler, Afrasiabi and McCauley (1991) who studied changes in energy input–output coefficients. There are many other energy-related studies in which Casler is a contributor.

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11}(\text{BTU/BTU}) & \mathbf{L}_{12}(\text{BTU/\$}) \\ \mathbf{L}_{21}(\text{\$/BTU}) & \mathbf{L}_{22}(\text{\$/\$}) \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_e(\text{BTU}) \\ \mathbf{f}_{\$}(\text{\$}) \end{bmatrix}$$

So $\mathbf{x} = \begin{bmatrix} \mathbf{x}_e \\ \mathbf{x}_{\$} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$, where $\underset{(k \times 1)}{\mathbf{x}_e}$ represents output of the energy sectors and $\underset{[(n-k)\times 1]}{\mathbf{x}_{\$}}$ is a vector of outputs of the other sectors. In particular, $\mathbf{x}_e = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$. For example, to assess the influence on energy sectors of changing final demands, the authors use

$$\Delta \mathbf{x}_e^{1985/1972} = [(\mathbf{L}_{11}^{1985})\mathbf{f}_1^{1985} + (\mathbf{L}_{12}^{1985})\mathbf{f}_2^{1985}] - [(\mathbf{L}_{11}^{1985})\mathbf{f}_1^{1972} + (\mathbf{L}_{12}^{1985})\mathbf{f}_2^{1972}]$$

Further decompositions into final-demand level and mix and technology change in energy and in non-energy inputs are found following the framework in earlier sections of this chapter. There is an interaction term at each decomposition level. The authors admit that there is no consistent set of guidelines regarding what to do with it, so they just report it as a separate component.[24]

2. Rose and Chen (1991). This study is also concerned with changes in energy use. Here final-demand contributions were also broken down into level and mix, and changes in technology were decomposed into a large number of either individual or interactive effects involving capital (K), labor (L), energy (E), and materials (M), along the lines of a two-tier KLEM production function. Coal, petroleum, natural gas, and electricity are examined separately. (There were 14 change components in all.)
3. Lin and Polenske (1995). This study focuses on changes in energy use in China over 1981–1987. The usual input–output accounting equation $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$ is accompanied by the energy accounting identity $\mathbf{E} = \mathbf{E}_g + \mathbf{E}_d$ [total energy consumption equals intermediate energy consumption (used in production activities) plus final energy consumption]. This is expressed as $\mathbf{m}\mathbf{x} = \mathbf{m}\mathbf{A}\mathbf{x} + \mathbf{m}\mathbf{f}$, where $\mathbf{m}$ is created from an identity matrix by keeping a 1 only in those column locations that correspond to energy sectors; that is, $\mathbf{m}$ selects the energy rows in $\mathbf{A}\mathbf{x}$, $\mathbf{f}$, and $\mathbf{x}$. This approach was already noted, above, in Roy, Das and Chakraborty (2002), but the Lin and Polenske study precedes that. [This is an alternative to rearranging (renumbering) sectors so that the energy sectors are all together – for example, the first k, as in the OTA study,

[24] They cite (a) Wolff (1985), who ignores it; (b) Feldman, McClain and Palmer (1987) and others, who allocate it equally among the other sources of change; and (c) Casler and Hannon (1989) and others, who "treat it separately and report its magnitude" (US Congress, OTA, 1990, p. 56).

above.] A little algebra shows that

$$\mathbf{E}_g = \mathbf{mAx} = \mathbf{m}[(\mathbf{I} - \mathbf{A})^{-1} - \mathbf{I}]\mathbf{f}$$

Decompositions are then carried out in the usual way.

4. Wier (1998). The concern of this study is to identify environmental effects of production, in particular the sources of emissions of carbon dioxide (CO_2), sulphur dioxide (SO_2), and nitrogen oxide (NO_x) in the Danish economy between 1966 and 1988, using a 117-sector input–output model for that country. The decompositions identify the following contributors: changes in energy intensity, changes in fuel-mix in production sectors, changes in fuel-mix in energy production sectors, input coefficient change (the $\mathbf{A}$ matrix), changes in final-demand level, and final-demand mix.
5. Kagawa and Inamura (2001). This model, for Japan, is in commodity-by-commodity format (Chapter 5), so the defining equation takes the form $\mathbf{q} = (\mathbf{I} - \mathbf{BC}^{-1})^{-1}\mathbf{e}$, relating commodity final demand to commodity output. Changes in total energy requirements over 1985–1990 are analyzed. Partitioned matrices are used (as in OTA, 1990) to distinguish between energy-supplying industries and other industries. The commodity technology assumption (where the simple technical coefficients matrix, $\mathbf{A}$, is replaced by $\mathbf{BC}^{-1}$) allows for an additional decomposition into both $\Delta\mathbf{B}$ and $\Delta\mathbf{C}^{-1}$ components (for both energy-supplying and non-energy sectors) – thereby reflecting changes in input structure and in product mix, respectively.

Many of the studies noted in Tables 13.6 and 13.7 were published in *Economic Systems Research*, and often they contain, in their references, a number of additional examples of SDA applications to which the interested reader can turn. It is important to remember that the figures presented in these tables are aggregates over all (frequently very many) sectors, or a subset of sectors, and of course all the rich sectoral detail is lost in such summary measures.[25] In general, analysts will often be interested in the more detailed results for individual sectors (or groups of sectors). This is the reason for including the detail on range of values in the Dietzenbacher and Hoekstra (2002) study in Table 13.6, where each of the figures in the table is an average over 25 values.

It also should be noted that percentage figures (as in these tables) are extremely sensitive to the differences between various changes. When a large positive effect (for example, final demand contribution) is nearly offset by a large negative effect (for example, technology change contribution), the percentages can be enormous. A simple table with several hypothetical results illustrates this fairly obvious fact.

Studies Using a Single-Region or Connected-Region Model

Washington State. Holland and Cooke (1992) used the structural decomposition framework at a regional (state) level to study the sources of change in the economy of Washington over 1963–1982, using the survey-based Washington input–output tables

[25] Most of the numbers in the tables were obtained by (simple) averaging over the more disaggregated results – either in value terms or in percentages – presented in the studies.

Table 13.7 Selected Empirical Structural Decompositions of Changes in Energy Use or Pollution Emissions

Author(s) and Source	Details (country; dates; changed variable(s); aggregation level)	Decomposition Components (percentage of total change[a]): Technology	Decomposition Components (percentage of total change[a]): Final Demand
US Congress, OTA (1990, Tabs. 2, 3, 6)	US; 1972–85; Δ (primary energy use); 88 sectors	−975 *Energy inputs* −770; *Non-energy inputs* −185; *Interaction* −20	720[b] *Level* 885; *Mix* −290; *Interaction* 125
Rose and Chen (1991)	US; 1972–82; Δ (energy use); 80 sectors	Coal, 64; Petroleum, 231; Natural gas, 65; Electricity, 56	Coal, 9; Petroleum, −370; Natural gas, −50; Electricity, 65[c] *Level*: Coal, 60; Petr.,−520; Nat. gas, −92; Elec., 70 *Mix*: Coal, −51; Petr., 150; Nat. gas, 42; Elec., −5
Lin and Polenske (1995, Table 3)	China; 1981–87; Δ (energy use); 18 sectors	−85 *Energy inputs* −106; *Non-energy inputs* 21	185 *Level* 196; *Mix* 3; *Distribution* −13
Wier (1998, Tables 3–5)	Denmark; 1966–88; Δ (pollution emissions); 117 sectors	−53 (CO_2) 373 (SO_2) 6 (NO_x)	153 (CO_2); −274 (SO_2); 95 (NO_x) *Level*: 175 (CO_2); −308 (SO_2) 112 (NO_x) *Mix*: −22 (CO_2); 34 (SO_2) −17 (NO_x)
Kagawa and Inamura (2001, Table 5)	Japan; 1985–90; Δ (total energy requirements); 94 sectors	0 *Energy inputs* 4; *Non-energy inputs* −4	100 *Energy* 8; *Non-energy* 92[d]

[a] Figures may not add to 100 percent due to rounding.
[b] Technology plus final-demand figures do not add to 100 percent because an interaction term between those two components is included in this study; in this case the interaction term is not small at 355%.
[c] Again, technology plus final-demand figures do not add to 100 percent because of an interaction term between the two components. This term is: Coal, 27; Petroleum, 39; Natural gas, 15; Electricity, −21.
[d] This figure is further decomposed into the following percentages: Household consumption (49), Non-household consumption (3), Capital formation, public (10), Capital formation, private (52); Other (−22).

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Cambridge, , GBR: Cambridge University Press, 2009. p 616.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=650

Table 13.8 SDA Percentage Change Sensitivities

Technology Change	Final-Demand Change	Total Change	Technology Change as a Percentage of Total Change	Final-Demand Change as a Percentage of Total Change
−50	51	1	−5000	5100
−50	52	2	−2500	2600
−48	52	4	−1200	1300
−55	45	10	−550	450

for 1963 and 1982. Reflecting a concern with the importance of trade for the Washington economy, they separated out the role of changes in demand (intermediate and final) within the state, within the rest of the USA (national markets), and outside the USA (international markets).

The US Multiregional Model (Miller and Shao, 1994). Two implementations of a multiregional input–output (MRIO) model for the US economy are available – for 1963 ($t = 0$) and 1977 ($t = 1$). The 1963 model takes the form

$$\mathbf{x}^0 = (\mathbf{I} - \mathbf{C}^0\mathbf{A}^0)^{-1}\mathbf{C}^0\mathbf{f}^0$$

and the 1977 model is

$$\mathbf{x}^1 = (\mathbf{I} - \mathbf{D}^1\mathbf{C}^1\mathbf{B}^1)^{-1}\mathbf{C}^1\mathbf{f}^1$$

The $\mathbf{C}^0$ and $\mathbf{C}^1$ matrices contain the interregional trade proportions for the two years. However, matrices $\mathbf{D}^1$ and $\mathbf{B}^1$ reflect technology in the 1977 model (only), which is based on commodity–industry input–output accounting.[26] Similarly, $\mathbf{A}^0$ is a matrix of technical coefficients in the 1963 model (only). Therefore, for simplicity, the superscripts on $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{A}$ can be eliminated, giving the following equation for gross output change over the period:

$$\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0 = (\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1\mathbf{f}^1 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0\mathbf{f}^0 \quad (13.35)$$

The two total requirements matrices (transforming final demands into outputs) can be denoted $\tilde{\mathbf{L}}^1 = (\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1$ and $\tilde{\mathbf{L}}^0 = (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0$.[27] Then

$$\Delta\mathbf{x} = \tilde{\mathbf{L}}^1\mathbf{f}^1 - \tilde{\mathbf{L}}^0\mathbf{f}^0 \quad (13.36)$$

This parallels (13.2), only now the two total requirements matrices are more complicated than the usual Leontief inverses, $\mathbf{L}^t = (\mathbf{I} - \mathbf{A}^t)^{-1}$. In particular, they incorporate both

[26] To be consistent with the 1963 model, in which industry final demands drive industry outputs, the 1977 model is in industry-by-industry format under the industry-based technology assumption.

[27] Appendix 13.1 indicates alternative ways of decomposing $\mathbf{x} = (\mathbf{I} - \mathbf{C}\mathbf{A})^{-1}\mathbf{C}\mathbf{f}$.

technology coefficients (**D** and **B** in one case, **A** in the other) and trade proportions ($\mathbf{C}^1$ and $\mathbf{C}^0$, respectively). In any event, following (13.7),

$$\Delta \mathbf{x} = (1/2)(\Delta \tilde{\mathbf{L}})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(\Delta \mathbf{f}) \tag{13.37}$$

where now $(\Delta \tilde{\mathbf{L}}) = \tilde{\mathbf{L}}^1 - \tilde{\mathbf{L}}^0$.

In (13.7) the two terms on the right captured the effects of technology change and final-demand change, respectively. Here, where

$$\Delta \tilde{\mathbf{L}} = (\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0 \tag{13.38}$$

the $(1/2)(\Delta \tilde{\mathbf{L}})(\mathbf{f}^0 + \mathbf{f}^1)$ term encompasses changes in both technology and trade.

Digging Deeper into $\Delta \tilde{\mathbf{L}}$: Technical Coefficients, Trade Structure

Decomposition 1. Create $\mathbf{M} = (\mathbf{I} - \mathbf{D}\mathbf{C}^0\mathbf{B})^{-1}\mathbf{C}^0$. This represents a kind of hybrid total requirements matrix that combines 1977 technology (in **B** and **D**) with 1963 trade structure (in $\mathbf{C}^0$). By subtracting and adding this term in (13.38),

$$\begin{aligned}\Delta \tilde{\mathbf{L}} = {} & [(\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{D}\mathbf{C}^0\mathbf{B})^{-1}\mathbf{C}^0] \\ & + [(\mathbf{I} - \mathbf{D}\mathbf{C}^0\mathbf{B})^{-1}\mathbf{C}^0 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0]\end{aligned} \tag{13.39}$$

The first term is a measure of the contribution to $\Delta \tilde{\mathbf{L}}$ made by changing trade proportions (with constant 1977 technology) and the second measures the effect on $\Delta \tilde{\mathbf{L}}$ of changing technology (with constant 1963 trade proportions). Then (13.39) can be written as

$$\Delta \tilde{\mathbf{L}} = \underbrace{(\tilde{\mathbf{L}}^1 - \mathbf{M})}_{\substack{\text{Trade change,} \\ \text{1977 technology}}} + \underbrace{(\mathbf{M} - \tilde{\mathbf{L}}^0)}_{\substack{\text{Technology change,} \\ \text{1963 trade patterns}}} \tag{13.40}$$

Decomposition 2. Consider, instead, $\mathbf{N} = (\mathbf{I} - \mathbf{C}^1\mathbf{A})^{-1}\mathbf{C}^1$. This is a kind of total requirements matrix that combines 1963 technology (in **A**) with 1977 trade structure (in $\mathbf{C}^1$). Subtracting and adding this term in (13.38) gives

$$\begin{aligned}\Delta \tilde{\mathbf{L}} = {} & [(\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{C}^1\mathbf{A})^{-1}\mathbf{C}^1] \\ & + [(\mathbf{I} - \mathbf{C}^1\mathbf{A})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0]\end{aligned} \tag{13.41}$$

In this case, the first term is a measure of the influence on $\Delta \tilde{\mathbf{L}}$ that is due to technology change (with constant 1977 trade proportions) and the second captures the effect on $\Delta \tilde{\mathbf{L}}$ of trade proportions change (assuming 1963 technology). Now, (13.41) can be written as

$$\Delta \tilde{\mathbf{L}} = \underbrace{(\tilde{\mathbf{L}}^1 - \mathbf{N})}_{\substack{\text{Technology change,} \\ \text{1977 trade patterns}}} + \underbrace{(\mathbf{N} - \tilde{\mathbf{L}}^0)}_{\substack{\text{Trade change,} \\ \text{1963 technology}}} \tag{13.42}$$

Averaging. Averaging the results in (13.40) and (13.42) in the usual way gives

$$\Delta\tilde{\mathbf{L}} = \underbrace{(1/2)(\tilde{\mathbf{L}}^1 + \mathbf{M} - \tilde{\mathbf{L}}^0 - \mathbf{N})}_{\text{Technology change effect}} + \underbrace{(1/2)(\tilde{\mathbf{L}}^1 + \mathbf{N} - \tilde{\mathbf{L}}^0 - \mathbf{M})}_{\text{Trade change effect}} \tag{13.43}$$

and, putting this result into (13.37)

$$\Delta\mathbf{x} = \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{M} - \tilde{\mathbf{L}}^0 - \mathbf{N})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{N} - \tilde{\mathbf{L}}^0 - \mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Trade change effect}} + \underbrace{(1/2)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(\Delta\mathbf{f})}_{\text{Final-demand change effect}} \tag{13.44}$$

Digging Deeper into $\Delta\mathbf{f}$: Level and Mix

The decompositions of $\Delta\mathbf{f}$ given in (13.20) – into level and mix – were also carried out. The final expression for $\Delta\mathbf{x}$ is

$$\Delta\mathbf{x} = \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{M} - \tilde{\mathbf{L}}^0 - \mathbf{N})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{N} - \tilde{\mathbf{L}}^0 - \mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Trade change effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(\Delta f)(\mathbf{B}^0 + \mathbf{B}^1)}_{\text{Final-demand level effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(f^0 + f^1)(\Delta\mathbf{B})}_{\text{Final-demand mix effect}} \tag{13.45}$$

This was used originally for a 70-sector, 51-region version of the model. The article presents results for a version aggregated to 10 sectors and nine regions. This means that in the original study there were 3,570 separate results for each of the decompositions. The results from this study noted in Table 13.9 below are averages over 90 outcomes for each decomposition. This illustrates again that a structural decomposition analysis for a reasonably large sized model generates an enormous amount of detail.

*A Multicountry Model for the European Community (*Oosterhaven and van der Linden, 1997*).* Here the authors are concerned with changes in value added that are associated with changes in output in a multicountry input–output setting. The model is a variant of the MRIO model, with 25 sectors, 8 countries and 4 categories of final demand in each country. Their decomposition follows the general structure of (13.34), with embellishments. Letting $\mathbf{v}^t$ and $\mathbf{c}^t$ represent column vectors of value added and value added per dollar of output at t, they work with

$$\mathbf{x}^t = \mathbf{L}^t\mathbf{f}^t = \mathbf{L}^t\mathbf{B}^t\mathbf{y}^t \text{ and } \mathbf{v}^t = \hat{\mathbf{c}}^t\mathbf{x}^t = \hat{\mathbf{c}}^t\mathbf{L}^t\mathbf{B}^t\mathbf{y}^t$$

(The bridge matrix, $\mathbf{B}^t$, and $\mathbf{y}^t$, which contains final-demand expenditures by final-demand category k in year t, were examined in section 13.1.2.)

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 619.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=653

Table 13.9 Selected Empirical Structural Decompositions at a Regional, Interregional or Multiregional Level

Author(s) and Source		Details Decomposition Components (percentage of total change[a]): Technology/Trade	Details Decomposition Components (percentage of total change[a]): Final Demand
Holland and Cooke (1992, Table 2)	Washington state; 1963–82; $\Delta\mathbf{x}$; 51 sectors	5	95 *Washington* 39 *Rest of US and world* 56
Miller and Shao (1994, Table 4)	US MRIO model; 1963–77; $\Delta\mathbf{x}$; 51 regions, 70 sectors; (aggregated to 9 regions, 10 sectors)	34 *Intraregional coefficients* 28 (19, 59)[b] *Interregional[b] coefficients* 6 (−43, 19)	67 *Level* 65 (53, 79) *Mix* 2 (−5, 13)
Oosterhaven and van der Linden (1997)	Intercountry model for EC; 1975–85; Δ (value added); 8 countries, 25 sectors	−2 *Intra-regional coeff.* 4 *Inter-regional coeff.* −2 *Value-added coeff.* −3	102 *Level* 102[c] *Mix* −1

[a] Figures may not add to 100 percent due to rounding.
[b] Figures in parentheses indicate the range of values across the nine regions in the study.
[c] This figure is further decomposed into the following percentages: Household consumption, 47; Government consumption, 20; Investment, 13; Exports to other EC countries, 9; Exports outside the EC, 12.

Then, following (13.12) for $n = 4$,

$$\Delta\mathbf{v} = (1/2)(\Delta\hat{\mathbf{c}})(\mathbf{L}^0\mathbf{B}^0\mathbf{y}^0 + \mathbf{L}^1\mathbf{B}^1\mathbf{y}^1) + (1/2)[(\hat{\mathbf{c}}^0)(\Delta\mathbf{L})(\mathbf{B}^1\mathbf{y}^1) + (\hat{\mathbf{c}}^1)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{y}^0)] + (1/2)[(\hat{\mathbf{c}}^0\mathbf{L}^0)(\Delta\mathbf{B})(\mathbf{y}^1) + (\hat{\mathbf{c}}^1\mathbf{L}^1)(\Delta\mathbf{B})\mathbf{y}^0)] + (1/2)(\hat{\mathbf{c}}^0\mathbf{L}^0\mathbf{B}^0 + \hat{\mathbf{c}}^1\mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{y}) \tag{13.46}$$

This accounts for the four components that contribute to the change in value added. The embellishments come from further decompositions of $\Delta\mathbf{L}$ and $\Delta\mathbf{B}$.

The European Union. The Dietzenbacher and Hoekstra study (Table 13.6) also has a spatial component because the data used came from intercountry input–output tables for the European Union (EU). This made it possible to disaggregate their final-demand component into: household consumption, other domestic final demands (government consumption, capital stock formation, inventory stock changes) and exports – to Germany, France, Italy, Belgium, Denmark, the rest of the EU, and the rest of the world.

Results from some of these studies are collected together in Table 13.9.

13.2 Mixed Models

In the usual form of the standard demand-side input–output model – $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$ and $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ – the final-demand elements, $\mathbf{f}$, are the exogenous components. Changes in the f_j come about as a result of forces that are outside the model (e.g., changes in consumer tastes, government purchases), and it is the effects of these changes on the economy's gross outputs, $\mathbf{x}$, that are quantified through the input–output model.

In certain situations a mixed type of input–output model may be appropriate, in which final demands for some sectors and gross outputs for the remaining sectors are specified exogenously. For example, due to a strike of a major supplier, output from a particular sector might be fixed at the amounts currently on hand in warehouses, awaiting transportation and delivery to buyers. Or, in a planned economy, a target might be to increase agricultural output by 12 percent by the end of the next planning period.

Mixed input–output models have often been applied in empirical studies in agricultural and resource economics. Some examples (discussed in Steinback, 2004) are:

- Agriculture [Johnson and Kulshreshtha, 1982 (economic importance of different farm types); Findeis and Whittlesey, 1984 (impacts of two irrigation development projects); Tanjuakio, Hastings and Tytus, 1996 (contribution of agriculture to the Delaware economy); Papadas and Dahl, 1999 (relative importance of 16 different US farm commodities); Roberts, 1994 (effects of milk production quotas)],
- Mining [Petkovich and Ching, 1978 (effects of partial elimination of mining in Nevada due to ore depletion)],
- Forestry [Eiser and Roberts, 2002 (relative economic importance of four different woodland types)],
- Fisheries [Leung and Pooley, 2002 (impacts of reduction in fishing areas in order to protect certain turtle populations)].

Most of these contain references to numerous additional studies.

All of the analysis in what follows is equally valid if we wish to model exogenous *changes in* some final demands and *changes in* gross outputs of the remaining sectors – that is, if the model is represented in $\Delta\mathbf{f}$ and $\Delta\mathbf{x}$ terms. We illustrate both scenarios below.

13.2.1 Exogenous Specification of One Sector's Output

Rearranging the Basic Equations As an example, in a three-sector model, assume that f_1, f_2, and x_3 are treated as exogenous. (Since the numbering of sectors is

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Cambridge, , GBR: Cambridge University Press, 2009. p 621.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=655

arbitrary, we can always assume that sector n is the one whose output, not final demand, is fixed.) The basic input–output relationships are still embodied in the following three equations:

$$
\begin{aligned}
(1-a_{11})x_1 - a_{12}x_2 - a_{13}x_3 &= f_1 \\
-a_{21}x_1 + (1-a_{22})x_2 - a_{23}x_3 &= f_2 \\
-a_{31}x_1 - a_{32}x_2 + (1-a_{33})x_3 &= f_3
\end{aligned}
$$

Rearrange all three equations in order to have the exogenous variables ($f_1, f_2,$ and x_3) on the right-hand side and the endogenous variables ($x_1, x_2,$ and f_3) on the left. This gives

$$
\begin{aligned}
(1-a_{11})x_1 - a_{12}x_2 + 0f_3 &= f_1 + a_{13}x_3 \\
-a_{21}x_1 + (1-a_{22})x_2 + 0f_3 &= f_2 + a_{23}x_3 \\
-a_{31}x_1 - a_{32}x_2 - f_3 &= -(1-a_{33})x_3
\end{aligned}
$$

It is clear that not only f_1 but now also $a_{13}x_3$ (for a fixed x_3) serve as exogenous "demand" for sector 1 (first equation) and similarly both f_2 and $a_{23}x_3$ are now exogenous drivers for sector 2. To facilitate later generalization, we rewrite these equations to include all variables in each equation. This gives

$$
\begin{aligned}
(1-a_{11})x_1 - a_{12}x_2 + 0f_3 &= f_1 + 0f_2 + a_{13}x_3 \\
-a_{21}x_1 + (1-a_{22})x_2 + 0f_3 &= 0f_1 + f_2 + a_{23}x_3 \\
-a_{31}x_1 - a_{32}x_2 - f_3 &= 0f_1 + 0f_2 - (1-a_{33})x_3
\end{aligned}
$$

In matrix form (we use partitioned matrices and vectors to emphasize differences from the standard input–output model) these two equations are

$$
\begin{bmatrix} (1-a_{11}) & -a_{12} & 0 \\ -a_{21} & (1-a_{22}) & 0 \\ -a_{31} & -a_{32} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \\ -(1-a_{33})x_3 \end{bmatrix} \tag{13.47}
$$

and

$$
\begin{bmatrix} (1-a_{11}) & -a_{12} & 0 \\ -a_{21} & (1-a_{22}) & 0 \\ -a_{31} & -a_{32} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & -(1-a_{33}) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.48}
$$

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 622.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=656

Let $\mathbf{M} = \begin{bmatrix} (1-a_{11}) & -a_{12} & 0 \\ -a_{21} & (1-a_{22}) & 0 \\ -a_{31} & -a_{32} & -1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & -(1-a_{33}) \end{bmatrix}$. Then (13.47) and (13.48) can be expressed as

$$\mathbf{M}\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \\ -(1-a_{33})x_3 \end{bmatrix} \tag{13.49}$$

and

$$\mathbf{M}\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \mathbf{N}\begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.50}$$

with solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \\ -(1-a_{33})x_3 \end{bmatrix} \tag{13.51}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.52}$$

Using results on partitioned matrix inverses (Appendix A), it can be shown that

$$\mathbf{M}^{-1} = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} & 0 \\ l_{21}^{(2)} & l_{22}^{(2)} & 0 \\ \beta_1 & \beta_2 & -1 \end{bmatrix}$$

where $\mathbf{L}^{(2)} = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} (1-a_{11}) & -a_{12} \\ -a_{21} & (1-a_{22}) \end{bmatrix}^{-1}$, the Leontief inverse for a *two-sector* model.[28] The important result to notice is that the inverse of the smaller model is a component in $\mathbf{M}^{-1}$. Carrying out the multiplication $\mathbf{M}^{-1}\mathbf{N}$, (13.52) is

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(2)} & \mathbf{L}^{(2)}\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} & \gamma \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.53}$$

[28] Here and in Appendix 13.2 we will sometimes find it helpful to use $\mathbf{A}^{(k)}$ and $\mathbf{L}^{(k)} = (\mathbf{I} - \mathbf{A}^{(k)})^{-1}$ to identify coefficient and Leontief inverse matrices for a k-sector input–output model.

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Cambridge, , GBR: Cambridge University Press, 2009. p 623.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=657

The exact values of β_1, β_2, and γ need not concern us at this point.

Of particular interest is the result for the endogenous outputs, x_1 and x_2,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} x_3 = \mathbf{L}^{(2)} \begin{bmatrix} f_1 + a_{13} \\ f_2 + a_{23} \end{bmatrix} x_3 \tag{13.54}$$

Suppose that a decision has been made to increase sector 3 output to some amount, $\bar{x}_3$, for whatever reason (for example, to fill back orders, or because of *anticipated* new demand, etc.). Using (13.54), we have $f_1 = 0$, $f_2 = 0$, and $x_3 = \bar{x}_3$, and the effects on sectors 1 and 2 are found as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3 = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3 \tag{13.55}$$

The vector $\begin{bmatrix} a_{13}\bar{x}_3 \\ a_{23}\bar{x}_3 \end{bmatrix}$ translates the new sector 3 output into sector 3's increased demands for inputs from sectors 1 and 2, and the inverse for the two-sector model converts these input demands into total necessary gross outputs from those two sectors.

"Extracting" the Sector There is an alternative approach that leads to precisely the same algebraic results for the impact of $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and x_3 on $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If we modify the $\mathbf{A}$ matrix for the three-sector model by setting all the coefficients in row 3 equal to zero – $\tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$ – we generate $(\mathbf{I} - \tilde{\mathbf{A}}) = \begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ 0 & 0 & 1 \end{bmatrix}$ and, importantly,[29]

$$(\mathbf{I} - \tilde{\mathbf{A}})^{-1} = \begin{bmatrix} \mathbf{L}^{(2)} & \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

This result depends, again, on properties of inverses to partitioned matrices (Appendix A). It is explored further in Appendix 13.2 to this chapter.

Consequently,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(2)} & \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix}$$

and the results for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are identical to those in (13.53) or (13.54). (We show in Appendix 13.2 that this approach is valid for the general case of $n - k$ sectors with exogenized outputs.)

29 Notice that although $\tilde{\mathbf{A}}$ is singular (a row of all zeros), $(\mathbf{I} - \tilde{\mathbf{A}})$ is not; and it is the latter matrix whose inverse is needed.

In a regional context, this approach seems to have first been discussed in Tanjuakio, Hastings and Tytus (1996); it is also featured in Steinback (2004). The economic logic, at the regional level, is that the regional purchase coefficients for the exogenized sectors are set equal to zero, thereby creating zero rows in **A** and eliminating those sectors as suppliers of (additional) interindustry inputs. It may be particularly helpful in regional situations where the **A** matrix of a ready-made input–output model is available (e.g., IMPLAN) and can be easily altered by zeroing out appropriate rows.[30]

13.2.2 An Alternative Approach When $f_1, \ldots, f_{n-1}$ and x_n Are Exogenously Specified[31]

This alternative makes use of the concept of an "output-to-output" multiplier (section 6.5.3). Recall that $\mathbf{L}^* = [l_{ij}^*] = \mathbf{L}\hat{\mathbf{L}}^{-1}$, where

$$l_{ij}^* = l_{ij}/l_{jj} = [\Delta x_i/\Delta f_j]/[\Delta x_j/\Delta f_j] = \Delta x_i/\Delta x_j$$

These elements, l_{ij}^*, are viewed as "output–to–output" multipliers. Each of the elements in column j of $\mathbf{L}^*$ indicates the amount of sector i output (the row label) that would be required if the *output* of sector j were one dollar.

If sector j increases its output to some new amount, $\bar{x}_j$, then $\mathbf{L}^*\bar{\mathbf{x}}$ (where $\bar{\mathbf{x}} = [0, \ldots, 0, \bar{x}_j, 0, \ldots, 0]'$) will generate a vector of total new outputs necessary from each sector in the economy because of the exogenously determined output in sector j. That is,

$$\mathbf{x}^* = \mathbf{L}^*\bar{\mathbf{x}} \tag{13.56}$$

This calculation gives the same result for the endogenous x_i as found using the approach in (13.55), above, as is demonstrated in the following examples. This result is shown to hold for the general case in Appendix 13.2.

The structure of $\mathbf{L}^*$ makes clear that a standard Leontief inverse, $\mathbf{L}$, can easily be used to capture impacts when any sector's output is made exogenous. If the output of sector j is specified exogenously, then all that is needed is that the elements in column j of $\mathbf{L}$ (known) be divided by l_{jj} (known). Put otherwise, standard demand-driven output multipliers for sector j will *uniformly overestimate* output–to–output multipliers for sector j by $[(l_{jj}-1)\times 100]$ percent. [The reader can easily show that $(l_{ij}-l_{ij}^*)/l_{ij}^* = l_{jj}-1$, given that $l_{ij}^* = l_{ij}/l_{jj}$.][32]

30 This approach is closely related to variants of the "hypothetical extraction" method for assessing a sector's importance to an economy through measures of sectoral "linkage" (a topic explored in section 12.2.5).

31 This approach is apparently first discussed in Evans and Hoffenberg (1952) and again in Ritz and Spaulding (1975, p. 14).

32 Roberts (1994) provides a numerical illustration in an empirical application with both standard output multipliers (final demand driven, from **L**) and those from $\mathbf{L}^*$ (output driven) for the case in which the milk sector's output is made exogenous. The (constant) percentage overestimation in the **L** model is 8.09, and the milk sector's on-diagonal element in **L** is 1.0809.

13.2.3 Examples with x_n Exogenous[33]

Suppose, as above, that we have a three-sector model in which f_1, f_2, and x_3 are treated as exogenous. Let $\mathbf{A} = \begin{bmatrix} .15 & .25 & .30 \\ .20 & .05 & .18 \\ .20 & .20 & .10 \end{bmatrix}$ (the first two rows and columns repeat the two-sector example in Chapter 2). In the format of (13.50), we have

$$\begin{bmatrix} .85 & -.25 & 0 \\ -.20 & .95 & 0 \\ -.20 & -.20 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & .30 \\ 0 & 1 & .18 \\ 0 & 0 & -.9 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix}$$

In particular,

$$\mathbf{M} = \begin{bmatrix} .85 & -.25 & 0 \\ -.20 & .95 & 0 \\ -.20 & -.20 & -1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & .30 \\ 0 & 1 & .18 \\ 0 & 0 & -.9 \end{bmatrix} \text{ and}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1.2541 & .3300 & 0 \\ .2640 & 1.1221 & 0 \\ -.3036 & -.2904 & -1 \end{bmatrix}$$

so

$$\mathbf{M}^{-1}\mathbf{N} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix}$$

Notice that the 2×2 upper-left submatrix in $\mathbf{M}^{-1}$ *and* in $\mathbf{M}^{-1}\mathbf{N}$ is indeed just the inverse of $\begin{bmatrix} .85 & -.25 \\ -.20 & .95 \end{bmatrix}$ from the two-sector model in Chapter 2:

$$\mathbf{L}^{(2)} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix}$$

Example 1: $f_1 = 100{,}000$, $f_2 = 200{,}000$, $x_3 = 150{,}000$ In this case, from (13.53),

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 256,750 \\ 293,000 \\ 25,050 \end{bmatrix}$$

[33] Even though we will use four figures to the right of the decimal in the numerical illustrations, comparisons of alternative techniques will still display small differences due to rounding, especially when matrices are inverted.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 626.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=660

If we are only interested in the effects on the gross outputs of sectors 1 and 2, then, from (13.53),

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} x_3 = \mathbf{L}^{(2)} \begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \end{bmatrix}$$

and for this example, $\begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} 145,000 \\ 227,000 \end{bmatrix}$, so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 145,000 \\ 227,000 \end{bmatrix} = \begin{bmatrix} 256,755 \\ 292,997 \end{bmatrix}$$

(The differences between these values and those for x_1 and x_2 in the three-sector version come about because of rounding in the computation of $\mathbf{M}^{-1}\mathbf{N}$, in particular in the elements in the third column of that matrix.)

Example 2: $f_1 = f_2 = 0$, $x_3 = 150,000$

Approach I. Suppose that only $x_3 = 150,000$ is exogenously specified; then $f_1 = f_2 = 0$, and (13.53) leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 65,340 \\ 42,180 \\ 113,490 \end{bmatrix}$$

Again, if only the gross outputs of sectors 1 and 2 are of interest, and since $f_1 = f_2 = 0$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 45,000 \\ 27,000 \end{bmatrix} = \begin{bmatrix} 65,345 \\ 42,177 \end{bmatrix}$$

(Differences are again due to rounded elements in the third column of $\mathbf{M}^{-1}\mathbf{N}$.)

Approach II. Continuing with the same numerical example but using the alternative approach, we create $\mathbf{L}^*$ for our three-sector illustration. Here,

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.4289 & .4973 & .5758 \\ .3769 & 1.2300 & .3716 \\ .4013 & .3838 & 1.3216 \end{bmatrix}$$

and so

$$\mathbf{L}^* = \mathbf{L}\hat{\mathbf{L}}^{-1} = \begin{bmatrix} 1 & .4043 & .4356 \\ .2637 & 1 & .2812 \\ .2808 & .3121 & 1 \end{bmatrix}$$

Consider again the case in which sector 3's output is set at \$150,000. Here, then,

$$\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 150,000 \end{bmatrix}$$

and, as in (13.56),

$$\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & .4043 & .4356 \\ .2637 & 1 & .2812 \\ .2808 & .3121 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 65,340 \\ 42,180 \\ 150,000 \end{bmatrix}$$

These values for x_1 and x_2 are the same as in our earlier results for the three-sector model (Approach I), and of course $x_3 = 150,000$, which is part of the stipulation of the problem and is assured by the fact that $l^*_{33} = 1$. (By definition, all $l^*_{jj} = 1$.)

In Appendix 13.2 we demonstrate that these two approaches for the case when x_n is exogenously specified must always give the same results for the outputs of x_1 through x_{n-1}. (Again, results on the inverse of a partitioned matrix turn out to be useful.)

Example 3: $f_1 = 100,000$, $f_2 = 200,000$, $x_3 = 100,000$ Consider the same three-sector model, with exogenous values $f_1 = 100,000$, $f_2 = 200,000$ (both as before), but $x_3 = 100,000$ (instead of 150,000). Using (13.53), we have

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ 100,000 \end{bmatrix} = \begin{bmatrix} 234,970 \\ 278,940 \\ -12,780 \end{bmatrix}$$

This simply means that the exogenously specified values of f_1, f_2, and x_3 in this example cannot possibly be satisfied unless f_3 is negative. If all variables represent *changes in* ($\Delta\mathbf{x}$ and $\Delta\mathbf{f}$) then to *increase* final demand for sectors 1 and 2 by 100,000 and 200,000, while *increasing* output of sector 3 by only 100,000, can only be accomplished by *decreasing* final demand for sector 3 by 12,780. This is not unusual in planned economies; increased *production* targets ($\Delta x_i > 0$) may be attainable only through decreases in allocations to *consumption* ($\Delta f_i < 0$). Similarly, in the case of a shortage (due to a strike, for example), increases in consumption in other sectors may require a decrease in consumption of the product that is in short supply. Whether or not negative values for f_3 make sense depends entirely on the context of the problem. If all x's and f's are *not* changes in, it may still be possible to attach meaning to a negative f_j. For example, if the exports component of final demand is defined as *net* exports, then a negative value here for f_j would mean net *imports* of j-type goods.

Example 4: The Critical Value of x_3 From the solution in (13.53), using the example values of $f_1 = 100,000$ and $f_2 = 200,000$, we can find the critical value of x_3 (call it $\bar{x}_3^c$) that makes $f_3 = 0$. (For x_3 above this value, f_3 will be positive; for x_3 below this value, f_3 will be negative.) Replacing 100,000 by $\bar{x}_3^c$ and setting $f_3 = 0$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ \bar{x}_3^c \end{bmatrix}$$

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 628.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=662

From the third equation, $0 = (-.3036)(100,000) + (-.2904)(200,000) + (.7566)\bar{x}_3^c$ or $\bar{x}_3^c = 116,891$.

Multipliers From the discussion thus far and from these numerical examples, we recognize that $\mathbf{M}^{-1}\mathbf{N}$ is a *multiplier matrix* that relates the exogenously given values, $\mathbf{x}^{ex} = [x_3]$ and $\mathbf{f}^{ex} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ in our examples, to those remaining x's and f's that are endogenous, $\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{f}^{en} = [f_3]$ in our examples. The elements in this matrix have the same kind of multiplier interpretation as we explored in Chapter 6 for the usual input–output system – $\mathbf{x} = \mathbf{L}\mathbf{f}$. In this example,

$$\mathbf{M}^{-1}\mathbf{N} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix}$$

So, for example, if $\Delta f_1 = 1$, $\Delta f_2 = \Delta x_3 = 0$, we see that $\Delta x_1 = 1.2541$, $\Delta x_2 = 0.2640$ and $\Delta f_3 = -0.3036$; if only final demand for sector 1 increases, then output in sectors 1 and 2 must increase while final demand for sector 3 goods must decrease. The elements in the second column have a similar interpretation. The third column contains elements that multiply changes in sector 3's *output* to generate consequent changes in outputs of sectors 1 and 2 (and final demand for sector 3). Specifically, $\Delta f_1 = \Delta f_2 = 0$, $\Delta x_1 = (.4356)\Delta x_3$ and $\Delta x_2 = (.2812)\Delta x_3$. These third column elements in $\mathbf{M}^{-1}\mathbf{N}$ are thus exactly the "output–to–output" multipliers that we created in deriving $\mathbf{L}^*$. Notice in particular (Example 3, above) that $l_{13}^* = 0.4356$ and $l_{23}^* = 0.2812$; these are precisely the elements in corresponding positions in $\mathbf{M}^{-1}\mathbf{N}$. This is no accident; Appendix 13.2 demonstrates why this will always be the case.

13.2.4 *Exogenous Specification of $f_1, \ldots, f_k, x_{k+1}, \ldots, x_n$*

The reader can easily work out the matrix representation of, say, a four-sector model with f_1, f_2, x_3, and x_4 exogenous, starting from the basic $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$ relationships to generate the parallel to, say, (13.48). For the general n-sector case, assume that sectors have been labeled so that the outputs of the first k sectors are endogenous:[34]

$$\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

[34] Sectors in an n-sector model can always be numbered so that the first k are those with endogenous gross outputs and the remaining $(n-k)$ have exogenous gross outputs.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 629.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=663

and the corresponding final demands are exogenous:

$$\mathbf{f}^{ex} = \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}$$

Similarly the last $(n-k)$ sectors are those whose gross outputs are exogenous:

$$\mathbf{x}^{ex} = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix}$$

and corresponding final demands are endogenous:

$$\mathbf{f}^{en} = \begin{bmatrix} f_{k+1} \\ \vdots \\ f_n \end{bmatrix}$$

Partition the coefficients matrix as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11} = \mathbf{A}^{(k,k)}$ denotes the submatrix made up of the first k rows and the first k columns of $\mathbf{A}$ [this can also be denoted $\mathbf{A}^{(k)}$ (see footnote 24)], $\mathbf{A}_{12} = \mathbf{A}^{[k,-(n-k)]}$ denotes the submatrix made up of the *first k rows* and the *last* $(n-k)$ *columns* of $\mathbf{A}$, $\mathbf{A}_{21} = \mathbf{A}^{[-(n-k),k]}$ denotes the submatrix made up of the *last* $(n-k)$ *rows* and the *first k columns* of $\mathbf{A}$, and $\mathbf{A}_{22} = \mathbf{A}^{[-(n-k),-(n-k)]}$ denotes the submatrix containing the *last* $(n-k)$ *rows and columns* of $\mathbf{A}$, and where the $\mathbf{I}$ and $\mathbf{0}$ matrices are of appropriate dimension in each case. The notation in the last three cases is necessary in order to distinguish *specific* row and column composition of a matrix from the general notation $\mathbf{A}^{(k)}$ for the coefficient matrix of a k-sector input–output model.

The generalization of (13.48) for $(n-k)$ exogenous outputs is

$$\begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix} \tag{13.57}$$

The solution procedure is the same as for any square set of linear equations. Using the same notation as earlier, in the case when only x_n was exogenous, we have $\mathbf{M} = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix}$, so the solution to $\mathbf{M}\begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} =$

$\mathbf{N}\begin{bmatrix}\mathbf{f}^{ex}\\ \mathbf{x}^{ex}\end{bmatrix}$, namely $\begin{bmatrix}\mathbf{x}^{en}\\ \mathbf{f}^{en}\end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix}\mathbf{f}^{ex}\\ \mathbf{x}^{ex}\end{bmatrix}$, becomes[35]

$$\begin{bmatrix}\underset{(k\times 1)}{\mathbf{x}^{en}}\\ \underset{[(n-k)\times 1]}{\mathbf{f}^{en}}\end{bmatrix} = \begin{bmatrix}\mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12}\\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & (\mathbf{I}-\mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12}\end{bmatrix}\begin{bmatrix}\underset{(k\times 1)}{\mathbf{f}^{ex}}\\ \underset{[(n-k)\times 1]}{\mathbf{x}^{ex}}\end{bmatrix} \tag{13.58}$$

where $(\mathbf{I}-\mathbf{A}^{(k)})^{-1} = \mathbf{L}^{(k)}$. (We indicate the dimensions of exogenous and endogenous vectors as an aid for what follows.) The parallel result for the earlier case is in (13.53).

As a check on the logic of the results in (13.57), notice that the two "extreme cases" correspond exactly to the basic input–output model.

Case 1: No exogenous outputs. Here $k = n$, $\mathbf{L}^{(k)} = \mathbf{L}^{(n)}$, $\mathbf{x}^{en} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, $\mathbf{f}^{ex} = \begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix}$, and $\mathbf{A}_{21}, \mathbf{A}_{12}, (\mathbf{I}-\mathbf{A}_{22}), \mathbf{x}^{ex}$, and $\mathbf{f}^{en}$ do not exist, so (13.57) is just the standard input–output model $(\mathbf{I}-\mathbf{A}^{(n)})\mathbf{x}^{en} = \mathbf{f}^{ex}$.

Case 2: All outputs exogenous. Here $k = 0, (\mathbf{I}-\mathbf{A}_{22}) = (\mathbf{I}-\mathbf{A}^{(n)}), \mathbf{x}^{ex} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, $\mathbf{f}^{en} = \begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix}$, and $\mathbf{L}^{(k)}, \mathbf{A}_{21}, \mathbf{A}_{12}, \mathbf{x}^{en}$, and $\mathbf{f}^{ex}$ disappear from (13.57), leaving $-\mathbf{I}\mathbf{f}^{en} = -(\mathbf{I}-\mathbf{A}^{(n)})\mathbf{x}^{ex}$ or $(\mathbf{I}-\mathbf{A}^{(n)})\mathbf{x}^{ex} = \mathbf{f}^{en}$. In words, if you specify all n outputs in the standard model, the n final demands are uniquely determined.

Consider also the two "less extreme" cases, which make more sense when we are dealing with the model in "changes in" (Δ) form, namely in which either $\Delta\mathbf{x}^{ex} = \mathbf{0}$ or $\Delta\mathbf{f}^{ex} = \mathbf{0}$.

Case 3: $\Delta\mathbf{x}^{ex} = \mathbf{0}$. Here the model is driven only by changes in final demands for sectors $1, \ldots, k$; $\Delta\mathbf{f}^{ex} \neq \mathbf{0}$. Then $\Delta\mathbf{x}^{en} = \mathbf{L}^{(k)}\Delta\mathbf{f}^{ex}$, which is a standard k-sector input–output model. As a consequence, $\Delta\mathbf{f}^{en} = -\mathbf{A}_{21}\mathbf{L}^{(k)}\Delta\mathbf{f}^{ex} = -\mathbf{A}_{21}\Delta\mathbf{x}^{en}$. This is a perfectly logical but not very interesting case. Here if $\Delta\mathbf{f}^{ex} > \mathbf{0}$, then $\Delta\mathbf{x}^{en} \geq \mathbf{0}$ and $\Delta\mathbf{f}^{en} = -\mathbf{A}_{21}\Delta\mathbf{x}^{en} \leq \mathbf{0}$, meaning that the changes in at least some of the $n-k$ endogenous final demands are necessarily negative. In words, since $\Delta\mathbf{x}^{ex} = \mathbf{0}$, the needs of sectors $1, \ldots, k$ for inputs from sectors $k+1, \ldots, n$, as itemized in $\mathbf{A}_{21}$, must be met by reductions in the amounts available for final demands for those remaining sectors.

Case 4: $\Delta\mathbf{f}^{ex} = \mathbf{0}$. Here the model is driven only by changes in outputs of sectors $k+1, \ldots, n$; $\Delta\mathbf{x}^{ex} \neq \mathbf{0}$. In this case,

$$\Delta\mathbf{x}^{en} = \mathbf{L}^{(k)}\mathbf{A}_{12}\Delta\mathbf{x}^{ex} \tag{13.59}$$

[35] This depends on results for the inverse of a partitioned matrix and also on the straightforward rules for multiplication of partitioned matrices. See Appendix A.

where $\mathbf{A}_{12} = \mathbf{A}^{[k,-(n-k)]} = \begin{bmatrix} a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k,k+1} & \cdots & a_{kn} \end{bmatrix}$. For example, the first element in $\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ will be $a_{1,k+1}\Delta x_{k+1} + a_{1,k+2}\Delta x_{k+2} + \cdots + a_{1n}\Delta x_n$; this represents the inputs that are needed from endogenous sector 1 to allow production of the fixed amounts of output in sectors $k+1, \ldots, n$. [In an early application of input–output analysis at the regional level, Tiebout (1969) specified (projected) the outputs of 13 out of 57 local sectors exogenously and found the consequent outputs of the remaining 44 sectors in the regional economy in just this way.]

At the same time, in this scenario $\Delta\mathbf{f}^{en} = [(\mathbf{I} - \mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12}]\Delta\mathbf{x}^{ex}$. This is exactly the structure as we saw earlier in examining interregional feedback effects in an interregional input–output model (Chapter 3) and in multiplier decompositions (Chapter 6). Here the logic is essentially the same: (a) $\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ identifies inputs from endogenous sectors to satisfy $\Delta\mathbf{x}^{ex}$; (b) $\mathbf{L}^{(k)}\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ converts those needs into *total* endogenous sector production (direct plus indirect effects); (c) $\mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ then translates that production into necessary inputs from exogenous sectors; and (d) since $\Delta\mathbf{x}^{ex}$ has already been fixed, this added amount must be netted out of what would have otherwise been available for final demands in sectors $k+1, \ldots, n$, $(\mathbf{I} - \mathbf{A}_{22})\Delta\mathbf{x}^{ex}$.

We will see in section 13.4 that a mix of x's and f's in the endogenous and exogenous categories can also be a useful framework for assessing the impact of a new industry on an economy.

13.2.5 An Example with x_{n-1} and x_n Exogenous

Example 5 (Example 2 expanded)

Approach I. Suppose now that both x_2 and x_3 are exogenous; along with $f_1 = 0$ and $x_3 = 150{,}000$ (as in Example 2), let $x_2 = 100{,}000$. Then, in terms of (13.57) (the reader might want, for practice, to check each of these submatrices as well as the subsequent matrix multiplications),

$$\mathbf{M} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{(1)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} .85 & 0 & 0 \\ -.2 & -1 & 0 \\ -.2 & 0 & -1 \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I} - \mathbf{A}_{22}) \end{bmatrix} = \begin{bmatrix} 1 & .25 & .3 \\ 0 & -.95 & .18 \\ 0 & .2 & -.9 \end{bmatrix}$$

giving

$$\begin{bmatrix} x_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.1765 & .2941 & .3530 \\ -.2535 & .8912 & -.2506 \\ -.2535 & -.2588 & .8294 \end{bmatrix} \begin{bmatrix} 0 \\ 100,000 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 82,360 \\ 51,530 \\ 98,530 \end{bmatrix} \qquad (13.60)$$

These results can be verified by noticing that $\begin{bmatrix} x_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 82,360 \\ 51,530 \\ 98,530 \end{bmatrix}$, along with $f_1 = 0, x_2 = 100,000$ and $x_3 = 150,000$, satisfy (except for rounding) the basic input–output equations, at the beginning of section 13.2.1.

Approach II. Alternatively, if we tried using $\mathbf{L}^*$, we would find

$$\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{L}^*\bar{\mathbf{x}} = \begin{bmatrix} 1 & .4043 & .4356 \\ .2637 & 1 & .2812 \\ .2808 & .3121 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 100,000 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 105,770 \\ 142,180 \\ 181,210 \end{bmatrix} \qquad (13.61)$$

which is totally wrong – neither x_2 nor x_3 is at its prespecified exogenous value and x_1 is wildly different from the result in (13.60). As already mentioned, we indicate in Appendix 13.2 why the $\mathbf{L}^*$ approach is only possible when just one sector's output is specified exogenously.

13.3 New Industry Impacts in the Input–Output Model

The input–output model provides a framework within which to assess the economic impact associated with the introduction of a new industry into an economy – for example, a basic manufacturing activity in a less-developed country, an export-oriented industry in a region, and so on. A quantitative approach to this kind of problem is extremely important. Individuals responsible for planning economic development (for a nation or a region) need to be able to make quantitative estimates of the total amount of economic benefit that can be expected from policies designed to attract certain kinds of industry to an area. Then the costs associated with attracting the activity – for example, reduced business taxes as an incentive, possible environmental degradation – can be weighed against the benefits of the new economic activity associated with the new industry. For convenience, in this section we will consider that the in-movement of the new industry is to a region, whether studied in isolation or as part of an interregional or multiregional system. It will be clear that the same principles apply if the "region" is in fact an entire country. In the input–output literature, one finds discussions of essentially two ways of introducing a new production activity into an economic area – through a new final-demand vector only and through the addition of new elements into the technical coefficients table for the economy. We examine these in turn.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 633.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=667

13.3.1 New Industry: The Final-Demand Approach

For illustration, we again consider a two-sector regional economy, for which we have a 2×2 input coefficient matrix, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If a firm in a different industry, which we will denote sector 3, were to locate in the region, one way of attempting to quantify the impact of this in-movement on the region is as follows.[36] From an input–output coefficient table for another region of the country, or from a national table, or from surveys, assume that it is possible to estimate what the inputs will be from sectors 1 and 2 per dollar's worth of output of the new sector 3; that is, a_{13} and a_{23}.

In order to quantify the impact of the in-movement of sector 3 to the economy, we must have some measure of the *magnitude* of new economic activity associated with sector 3. In input–output terms, this means that either sector 3's level of production (gross output), x_3, or of sales to final demand, f_3, must be specified. For this example, assume that the measure of new activity by sector 3 is gross output; denote this proposed level of sector 3 production by $\bar{x}_3$. This is often the case. A new firm plans to build, say, a \$2.5 million plant with a planned annual output of \$850,000, for example. Then the new demand on sectors 1 and 2 that arises because of production by the new sector 3 is $a_{13}\bar{x}_3$ and $a_{23}\bar{x}_3$, respectively. That is, we can view these new demands as an *exogenous* change imposed on the original two sectors; $\Delta\mathbf{f} = \begin{bmatrix} a_{13}\bar{x}_3 \\ a_{23}\bar{x}_3 \end{bmatrix}$, and so the impacts, in terms of the outputs from these two sectors, will be given by $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f}$:

$$\Delta\mathbf{x} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} a_{13}\bar{x}_3 \\ a_{23}\bar{x}_3 \end{bmatrix} = \begin{bmatrix} l_{11}a_{13}\bar{x}_3 + l_{12}a_{23}\bar{x}_3 \\ l_{21}a_{13}\bar{x}_3 + l_{22}a_{23}\bar{x}_3 \end{bmatrix} \tag{13.62}$$

Given that there are also the usual kinds of final demands, $\bar{f}_1$ and $\bar{f}_2$, for the products of the two sectors, total gross outputs in sectors 1 and 2 will be

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} \bar{f}_1 + a_{13}\bar{x}_3 \\ \bar{f}_2 + a_{23}\bar{x}_3 \end{bmatrix} = \begin{bmatrix} l_{11}(\bar{f}_1 + a_{13}\bar{x}_3) + l_{12}(\bar{f}_2 + a_{23}\bar{x}_3) \\ l_{21}(\bar{f}_1 + a_{13}\bar{x}_3) + l_{22}(\bar{f}_2 + a_{23}\bar{x}_3) \end{bmatrix} \tag{13.63}$$

This is exactly the structure of the model in (13.55), and for the same reason. We are specifying $\bar{f}_1$ and $\bar{f}_2$ and, in addition, the value of x_3. When $\bar{x}_3 = 0$, that is, without the new sector in the region, this is a standard input–output exercise. When $\bar{f}_1 = 0$ and $\bar{f}_2 = 0$, then in (13.63) we find the impact of the new industry alone – as in (13.62).

For example, using the same illustration, let $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$. Then $(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.253 & .330 \\ .264 & 1.122 \end{bmatrix}$. Assume that our estimates of the direct input coefficients for the new sector 3 are $a_{13} = 0.30$ and $a_{23} = 0.18$, and that the plant in the new sector 3 that is moving into the region expects to produce at a level of \$100,000 per year. So $\bar{x}_3 =$

[36] This is essentially the approach used by Isard and Kuenne (1953) and by Miller (1957) in early applications of the input–output framework at a regional level.

100,000, $\Delta\mathbf{f} = \begin{bmatrix} 30,000 \\ 18,000 \end{bmatrix}$, and, as in (13.62),

$$\Delta\mathbf{x} = \begin{bmatrix} 1.253 & .330 \\ .264 & 1.122 \end{bmatrix} \begin{bmatrix} 30,000 \\ 18,000 \end{bmatrix} = \begin{bmatrix} 43,560 \\ 28,116 \end{bmatrix} \tag{13.64}$$

Sector 1, in satisfying the new demand for \$30,000 worth of its product, will ultimately have to increase its output by \$43,560. Similarly, the new demands on sector 2 from sector 3 are \$18,000, but in the end sector 2 will need to produce a total of \$28,116 more output. These figures represent one way of measuring the impact on an economy that comes about from the in-movement of new industrial activity.

With a_{13} and a_{23} assumed known, but $a_{31} = a_{32} = a_{33} = 0$, the basic equations in this approach are

$$\begin{aligned} (1 - a_{11})x_1 - a_{12}x_2 - a_{13}x_3 &= f_1 \\ -a_{21}x_1 + (1 - a_{22})x_2 - a_{23}x_3 &= f_2 \\ 0x_1 + 0x_2 + x_3 &= f_3 \end{aligned}$$

The first two equations reflect the fact that sector 1 and 2 outputs are used as inputs to (the new) sector 3. The third equation shows that all of sector 3's output can be used to satisfy final demand, since it is not used as an input to production in the region. (For example, a sector may move to a region to be closer to the sources of inputs, while continuing to produce a product for export.)

In matrix terms, with

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } (\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} (1 - a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1 - a_{22}) & -a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

we have partially included the new sector in the $\mathbf{A}$ matrix. To assess the impact of new sector 3 production, $\bar{x}_3$, we let $f_1 = 0$ and $f_2 = 0$. Also, $f_3 = x_3 = \bar{x}_3$, from the third equation above. Thus

$$\mathbf{x} = \bar{\mathbf{L}} \begin{bmatrix} 0 \\ 0 \\ \bar{x}_3 \end{bmatrix}$$

where $\bar{\mathbf{L}} = [\bar{l}_{ij}] = (\mathbf{I} - \bar{\mathbf{A}})^{-1}$. Because of the zeros in $\mathbf{f}$, $x_1 = \bar{l}_{13}\bar{x}_3$, $x_2 = \bar{l}_{23}\bar{x}_3$, and $x_3 = \bar{l}_{33}\bar{x}_3$. That is, only the third column of the inverse is of interest. Using results on the inverse of a partitioned matrix (Appendix A) it is easily shown that

$$\begin{bmatrix} \bar{l}_{13} \\ \bar{l}_{23} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \text{ and } \bar{l}_{33} = 1$$

In particular, then,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3$$

exactly as in (13.62), above. Note also that, as expected, $x_3 = (1)\bar{x}_3$.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 635.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=669

13.3.2 New Industry: Complete Inclusion in the Technical Coefficients Matrix

The estimate of the impact of the new industry that was given above is clearly conservative; the complete impact of a new sector of an economy would reflect the fact that not only would the new industry buy inputs from existing sectors, but it would probably also sell its own product as an input to other producing sectors in the economy and ultimately the entire technical structure of the economy may change. In the first place, there will be a new column and row of direct-input coefficients associated with purchases by and sales of the new sector. In addition, there may be changes in the elements of the original **A** matrix, reflecting, for example, substitution of the newly available input for one previously used.

To completely "close" the previous 2×2 coefficient matrix with respect to the new industry, we need a_{13} and a_{23} (which we have already assumed can be estimated), and we also need a_{31} and a_{32}, estimates of how much each of the old industries (1 and 2) will buy from the new sector (3) per dollar's worth of their outputs, plus a_{33}, the intrasectoral input coefficient for the new industry. For in-movement of a new industry into a region with n original sectors, the previous approach required that we estimate n new coefficients (a column for the new sector, except for the last element). For the present approach we need an additional $(n+1)$ coefficients (a row for the new sector, including intraindustry use per dollar's worth of output); we need $(2n+1)$ new coefficients in all.

Again, assuming that x_3 is known, our three-equation model, relating the endogenous variables x_1, x_2, and f_3 to the values $\bar{f}_1$, $\bar{f}_2$, and $\bar{x}_3$, is still

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 - a_{13}\bar{x}_3 &= \bar{f}_1\\ -a_{21}x_1 + (1-a_{22})x_2 - a_{23}\bar{x}_3 &= \bar{f}_2\\ -a_{31}x_1 - a_{32}x_2 + (1-a_{33})\bar{x}_3 &= f_3\end{aligned} \tag{13.65}$$

Rearranging, to put exogenous variables on the right-hand side,

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 + 0f_3 &= \bar{f}_1 + a_{13}\bar{x}_3\\ -a_{21}x_1 + (1-a_{22})x_2 + 0f_3 &= \bar{f}_2 + a_{23}\bar{x}_3\\ -a_{31}x_1 - a_{32}x_2 - f_3 &= -(1-a_{33})\bar{x}_3\end{aligned} \tag{13.66}$$

The matrix representation for (13.66) is

$$\begin{bmatrix}(1-a_{11}) & -a_{12} & 0\\ -a_{21} & (1-a_{22}) & 0\\ -a_{31} & -a_{32} & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ f_3\end{bmatrix} = \begin{bmatrix}\bar{f}_1 + a_{13}\bar{x}_3\\ \bar{f}_2 + a_{23}\bar{x}_3\\ -(1-a_{33})\bar{x}_3\end{bmatrix} \tag{13.67}$$

This is exactly the structure of the model in (13.48), in the previous section, and so solution possibilities are the same as we saw in the examples of that section. In particular, there is no guarantee that the f_3 associated with given values $\bar{f}_1, \bar{f}_2$, and $\bar{x}_3$ will be positive.

Instead of specification of the level of gross output of the new sector, one could quantify the magnitude of the new operation by exogenously fixing the level of sales to final demand – that is, by specifying f_3 at $\bar{f}_3$, instead of x_3 at $\bar{x}_3$. But then, from (13.65), with x_3 now a variable to be determined (not specified at $\bar{x}_3$), we see that this is a standard kind of input–output problem. Whether $\bar{f}_1 = 0$ or not, and whether $\bar{f}_2 = 0$ or not, given some $\bar{f}_3 > 0$, we find the associated values of the necessary gross outputs, x_1, x_2, and x_3, through the use of the Leontief inverse to the 3×3 $(\mathbf{I} - \mathbf{A})$ matrix in (13.67). Thus, when the level of new sector activity is specified in terms of sales to final demand rather than gross output, no new principles are involved in assessing the impact on the economy into which the industry moves.

For example, for our illustrative problem, the 3×3 technical coefficients matrix is

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .225 & .30 \\ .20 & .05 & .18 \\ .20 & .20 & .10 \end{bmatrix}$$

(using an overbar to distinguish this from the original 2×2 $\mathbf{A}$ matrix). Thus the matrix of coefficients in the equations in (13.67) is

$$(\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} .85 & -.25 & -.30 \\ -.20 & .95 & -.18 \\ -.20 & -.20 & .90 \end{bmatrix} \tag{13.68}$$

and the corresponding inverse is

$$\bar{\mathbf{L}} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix} \tag{13.69}$$

Given $\bar{f}_1 = 100,000$, $\bar{f}_2 = 200,000$, and, say, $\bar{f}_3 = 50,000$, we find that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ 50,000 \end{bmatrix} = \begin{bmatrix} 271,100 \\ 302,300 \\ 183,000 \end{bmatrix} \tag{13.70}$$

in standard input–output fashion.

13.3.3 A New Firm in an Existing Industry

If the firm that moves into a region belongs to a sector that is already established in the region, so that the effect is to augment the production capacity of a particular existing industry, *not* introduce it into the local economy for the first time, the assessment of its impact is fairly straightforward. In particular, an input–output table for the economy in question will already include interindustry and intraindustry relationships for the sector in which the new firm is classified.

Assume that we have a three-sector economy and that the new firm is classified as a member of sector 3. Thus 3×3 $\mathbf{A}$ and $\mathbf{L}$ matrices are known. If the level of activity in the new firm is specified as a certain total amount of production, then we have a positive x_3^*,

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 637.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=671

and the relationships among sectors are exactly those shown in (13.65), above, where now we use x_3^* in place of $\bar{x}_3$ to distinguish the two cases ($\bar{x}_3$ when the industry was new to the region, x_3^* when the new firm only represents an increase in capacity of the existing sector). The new demands on the three original sectors are found as

$$\begin{bmatrix} a_{13}x_3^* \\ a_{23}x_3^* \\ a_{33}x_3^* \end{bmatrix} \tag{13.71}$$

and impacts on all three sectors are found in the standard input–output way:

$$\Delta\mathbf{x} = \mathbf{L}\begin{bmatrix} a_{13}x_3^* \\ a_{23}x_3^* \\ a_{33}x_3^* \end{bmatrix} \tag{13.72}$$

If the level of new capacity in sector 3 is specified through an additional amount of sales to final demand, that is, as Δf_3, then the impact is found in the usual input–output way. The new final-demand vector is $\begin{bmatrix} 0 \\ 0 \\ \Delta f_3 \end{bmatrix}$ and

$$\Delta\mathbf{x} = \mathbf{L}\begin{bmatrix} 0 \\ 0 \\ \Delta f_3 \end{bmatrix} \tag{13.73}$$

which is just

$$\Delta x_1 = l_{13}\Delta f_3, \Delta x_2 = l_{23}\Delta f_3, \Delta x_3 = l_{33}\Delta f_3 \text{ or } \Delta\mathbf{x} = \begin{bmatrix} l_{13} \\ l_{23} \\ l_{33} \end{bmatrix}(\Delta f_3) \tag{13.74}$$

For example, assume that the Leontief inverse for the three-sector economy is as shown in (13.69). If a new firm in sector 3 moves into the economy and its projected level of annual production is \$120,000 ($x_3^* = 120,000$), then, using the elements in the third column of the technical coefficients matrix, we find the new final demands in (13.71) as

$$\begin{bmatrix} (.30)(120,000) \\ (.18)(120,000) \\ (.10)(120,000) \end{bmatrix} = \begin{bmatrix} 36,000 \\ 21,600 \\ 12,000 \end{bmatrix}$$

and, as in (13.72)

$$\Delta\mathbf{x} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix}\begin{bmatrix} 36,000 \\ 21,600 \\ 12,000 \end{bmatrix} = \begin{bmatrix} 69,127 \\ 44,604 \\ 38,594 \end{bmatrix}$$

Notice that the *total* new output from sector 3 is \$158,594. This figure includes the \$120,000 from the new firm and \$38,594 of additional output from the old (existing)

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 638.
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firms in sector 3. On the other hand, if increased capacity in sector 3 is specified as, say, $70,000 more sales to final demand for sector 3 goods, then, as in (13.73),

$$\Delta\mathbf{x} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 70,000 \end{bmatrix} = \begin{bmatrix} 40,320 \\ 26,880 \\ 92,540 \end{bmatrix}$$

which is just $\begin{bmatrix} .576 \\ .372 \\ 1.322 \end{bmatrix}$ (70,000), as in (13.74).

13.3.4 Other Structural Changes

As already mentioned, when a new industry moves into an economic area, or when the capacity of an existing sector is increased, it is entirely possible that current transaction patterns for existing sectors in the region will change. For example, sector j, which formerly bought input i from a firm located outside the region, may now purchase some (or all) of input i from the new local establishment. Or, indeed, sector j may replace formerly used input k, bought from a producer in the region, with input i bought from the new establishment in the region. Such changes in transactions, the elements of the **Z** matrix, will generate changes in direct-input coefficients in columns and rows other than those for the new sector (or for the sector whose capacity has been increased.)

It should be clear that out-movement of a firm or an entire sector from a local economy can be treated in much the same way. Usually output, income, employment or value-added multipliers provide an adequate approach to quantifying such decreases in economic activity – particularly if, say, one plant closes but other plants in the same sector remain. If all economic activity in a sector is stopped – for example, all shoe manufacturing leaves Massachusetts and moves to the South – then the column and row for that sector disappear from the Massachusetts **A** matrix, and local producers in other sectors that use the product as an input will either have to import the good that has disappeared from the local economy or else they will substitute alternative locally produced inputs. Similarly, local firms that previously supplied inputs to the now-absent sector will find their sales patterns altered. Again, changes will occur in other columns and/or rows of the **A** matrix. However, it is extremely difficult to predict exactly where these changes will be and exactly what their magnitude will be.

13.4 Dynamic Considerations in Input–Output Models

13.4.1 General Relationships

Thus far, we have considered analysis using the **A** matrix of technical coefficients derived from measured *flows* of goods between sectors, purchased to serve current production needs during a particular period of time. Each of the flows, z_{ij}, is viewed as serving as an input for current output, x_j, and these relations are reflected in the technical coefficients, $a_{ij} = z_{ij}/x_j$. Actually, however, some input goods contribute to the production process but are not immediately used up during that production – machines, buildings, and so on. In other words, a sector has a certain capital stock that is also necessary for production. If one could measure the value of the output of sector i that is held by sector j as *stock*, k_{ij}, then one could estimate a "capital coefficient," by dividing this holding of stock by the output of sector j, over some period. Along with fixed investment items such as buildings and machinery, goods bought as inventory by sector j, to use as inputs to later production, may also be included in the k_{ij} term. Let $b_{ij} = k_{ij}/x_j$; this coefficient is interpreted as the amount of sector i's product (in dollars) held as capital stock for production of one dollar's worth of output by sector j.[37]

For example, if sector i is the construction industry and sector j is automobiles, b_{ij} might represent the dollars' worth of factory space per dollar's worth of automobiles produced. Clearly, for current production, the machinery, buildings, and so forth must already be in place. But if an economy is growing, then anticipated production (next year) is different from current production (this year), and the amount of supporting capital may change: one simple assumption (often used) is that the amount of new production from sector i for capital stocks in sector j in time period $t+1$ (say next year) will be given by $b_{ij}(x_j^{t+1} - x_j^t)$, where the superscripts denote time periods (here years); that is, the amount of sector i production necessary to satisfy the added demand in sector j for goods from sector i as capital stocks for next year's production is given by the observed capital coefficient, b_{ij}, times the change in sector j output between this year and next year, $(x_j^{t+1} - x_j^t)$. This use of the capital coefficients assumes that production is at or near effective capacity in sector j, since the anticipated increase in production, if $(x_j^{t+1} - x_j^t)$ is positive, requires new capital goods.[38]

The typical equation for the output of sector i in period t would become

$$x_i^t = \sum_{j=1}^{n} a_{ij}x_j^t + \sum_{j=1}^{n} b_{ij}(x_j^{t+1} - x_j^t) + f_i^t \qquad (13.75)$$

or

$$x_i^t - \sum_{j=1}^{n} a_{ij}x_j^t + \sum_{j=1}^{n} b_{ij}x_j^t - \sum_{j=1}^{n} b_{ij}x_j^{t+1} = f_i^t \qquad (13.76)$$

[37] It has become traditional to use b_{ij}, and later $\mathbf{B} = [b_{ij}]$, for capital coefficients in a dynamic input–output model. It is also traditional to use $\mathbf{B}$ in the Ghosh model, as we saw in section 12.1, and to represent a "bridge" matrix, as in section 13.1.8. The context should make clear which meaning is intended.

[38] The $x_j^{t+1} - x_j^t$ term could also be negative or zero. Thus, if $b_{ij} = 0.02$ and $x_j^{t+1} - x_j^t = \$100$, there will be a need for \$2 more output from sector i for sector j; if $x_j^{t+1} - x_j^t = -\$300$, the model would forecast a decrease of \$6 in purchases from i by j. In general, we are usually concerned with sectoral consequences of economic growth, so that the usual setting in which the dynamic model is used is when $x_j^{t+1} - x_j^t$ is strictly positive.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 640.
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The matrix form, using an $n \times n$ capital coefficients matrix $\mathbf{B} = [b_{ij}]$, is

$$(\mathbf{I} - \mathbf{A})\mathbf{x}^t - \mathbf{B}(\mathbf{x}^{t+1} - \mathbf{x}^t) = \mathbf{f}^t \text{ or } (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t - \mathbf{B}\mathbf{x}^{t+1} = \mathbf{f}^t \tag{13.77}$$

One rearrangement of this result is

$$\mathbf{B}\mathbf{x}^{t+1} = (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t - \mathbf{f}^t \tag{13.78}$$

for $t = 0, 1, \ldots, T$. For example, if the time superscripts denote years, this represents a set of relationships between gross outputs and final demands starting now (year $t = 0$) and extending T years into the future.[39]

These are linear *difference equations*, since the values of the variables – the x_j – are related for different periods of time via the coefficients in $\mathbf{A}$ and $\mathbf{B}$ and the final demands. Solution methods for sets of difference equations, and analysis of the values of the variables over time, are topics that go beyond the level of this text. The intention here is primarily to acquaint the reader with the notion of capital coefficients and with one of the ways in which the existence of stocks of capital goods for production have been incorporated into input–output analysis.[40] Clearly, the assumptions inherent in this model – for example, the stability of capital coefficients over time – deserve just as careful scrutiny as those in the static model. Moreover, data and measurement problems for estimating capital coefficients are even more severe than those for technical coefficients.

From (13.77) it is possible to derive either a "forward looking" or a "backward looking" expression. Solving for $\mathbf{x}^t$ in terms of $\mathbf{x}^{t+1}$ gives $\mathbf{x}^t = (\mathbf{I} - \mathbf{A} + \mathbf{B})^{-1}(\mathbf{B}\mathbf{x}^{t+1} + \mathbf{f}^t)$; letting $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$, this is $\mathbf{x}^t = \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^{t+1} + \mathbf{f}^t)$.[41] Each period's outputs depend on the outputs of the following period (and current period final demands). This kind of solution is possible as long as $\mathbf{G}^{-1}$ exists, and in practice $(\mathbf{I} - \mathbf{A} + \mathbf{B})$ is not likely to be singular. On the other hand, from (13.77) or (13.78) we can equally well find $\mathbf{x}^{t+1}$ as a function of $\mathbf{x}^t$, namely $\mathbf{x}^{t+1} = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^t - \mathbf{f}^t)$, and now each period's outputs depend on the outputs from the previous period (and, again, current period final demands). This approach requires that $\mathbf{B}$ be nonsingular, and, in fact, singularity of the $\mathbf{B}$ matrix is a problem in dynamic input–output models. It is easy to see why it might be that $|\mathbf{B}| = 0$. In a model with a fairly large number of sectors (a relatively disaggregated model), it is very likely that there will be sectors that do not supply capital goods to any sectors – that is, sectors whose row in the $\mathbf{B}$ matrix will contain all zeros. (For example, if there were a sector labeled "Agriculture, potatoes.") When one or more rows of a

[39] In some discussions of dynamic input–output models, the time superscripts are shifted "backward" by one period, leading to $(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^{t-1} - \mathbf{B}\mathbf{x}^t = \mathbf{f}^{t-1}$. There have also been differing labeling suggestions – "backward-lag" vs. "forward-lag" models – which need not concern us.

[40] For the reader who is familiar with differential calculus, there is a continuous version of this model. As the time interval between periods becomes very small, the difference $x_j^{t+1} - x_j^t$ approaches the derivative dx_j/dt. The continuous analog to (13.75) is thus $x_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}(dx_j/dt) + f_i$, and, denoting the time derivative of the vector $\mathbf{x}$ by $\dot{\mathbf{x}}$, we would have $\mathbf{B}\dot{\mathbf{x}} = (\mathbf{I} - \mathbf{A})\mathbf{x} - \mathbf{f}$. These are linear *differential equations* for which solution procedures and stability analysis are also possible but beyond the level of this text.

[41] Recall than $\mathbf{G}$ is also used in the Ghosh model, but the context should make clear which meaning is intended.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 641.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=675

matrix contains all zeros, the determinant of the matrix is zero and so the matrix has no inverse.[42] In later examples we will see that even when $\mathbf{B}$ is nonsingular, it may be somewhat "ill-conditioned" and contain unusually large elements in its inverse.

In developing capital coefficients, one may also wish to distinguish between "replacement capital" – for example, investment for replacing depreciated equipment – which is a function of current production, $\mathbf{x}^t$, and "expansion capital" – for example, investment in new equipment for expanded production capacity – which is a function of industry growth (the difference between current and past production, $\mathbf{x}^{t+1} - \mathbf{x}^t$). In this case we might write the analog to (13.77) as

$$(\mathbf{I} - \mathbf{A} - \mathbf{D} + \mathbf{B})\mathbf{x}^t - \mathbf{B}\mathbf{x}^{t+1} = \mathbf{f}^t$$

where $\mathbf{D}$ is the newly added matrix of replacement capital coefficients and $\mathbf{B}$ is now the matrix of expansion capital coefficients.

At a regional level, several operational models have been formulated, such as those found in Miernyk *et al.* (1970), which examines alternative economic development strategies for the state of West Virginia, and Miernyk and Sears (1974), where the impacts of pollution-control technologies on regional economies are analyzed, using a dynamic input–output model.

13.4.2 A Three-Period Example

Consider (13.77) again with $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$ and let $T = 3$. Then the difference equation relationships are

$$\begin{aligned} \mathbf{G}\mathbf{x}^0 - \mathbf{B}\mathbf{x}^1 &= \mathbf{f}^0 \\ \mathbf{G}\mathbf{x}^1 - \mathbf{B}\mathbf{x}^2 &= \mathbf{f}^1 \\ \mathbf{G}\mathbf{x}^2 - \mathbf{B}\mathbf{x}^3 &= \mathbf{f}^2 \\ \mathbf{G}\mathbf{x}^3 - \mathbf{B}\mathbf{x}^4 &= \mathbf{f}^3 \end{aligned}$$

or

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.79}$$

Notice that there are four matrix equations involving five unknown vectors, $\mathbf{x}^0$ through $\mathbf{x}^4$. If there are n sectors in the economy, we have $4n$ linear equations in $5n$ variables. An issue that arises in many dynamic models, including the input–output system, is which values to specify as fixed in the dynamic process. Generally, there are *initial* values, at the beginning ($t = 0$), when one starts with a given amount of, say,

[42] There is a large literature on singularity in the dynamic Leontief model and variations in the model that attempt to avoid the problem. This subject is vast and beyond the scope of this book. An interested reader might want to refer to Leontief (1970), Duchin and Szyld (1985), Leontief and Duchin (1986) or to Steenge and Thissen (2005) for critical summaries of many of these attempts to avoid or counteract the singularity problem.

output in the economy, or else there are *terminal* values, specifying desired characteristics of the system at the end of the period over which the model is being used ($t = T$ or $T + 1$). We investigate several possibilities in the case where $T = 3$.

Terminal Conditions In (13.79), when $T = 3$, this means $\mathbf{x}^{T+1} = \mathbf{x}^4$. In some versions of the dynamic input–output model (for example, Leontief, 1970), it is simply assumed that we cannot (or don't care to) see beyond year T; it is the last year that is of interest, and so $\mathbf{x}^{T+1} = \mathbf{0}$.[43] In that case, the equations in (13.79) become

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.80}$$

Since $\mathbf{x}^4 = \mathbf{0}$, it disappears from the $\mathbf{x}$ vector in (13.79), and the last column of the coefficient matrix in (13.79) is also unnecessary.

Given a set of final demands in the current year and in the next three years – $\mathbf{f}^0, \mathbf{f}^1, \mathbf{f}^2$, and $\mathbf{f}^3$ – we could find the associated gross outputs in each of those years – $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2$, and $\mathbf{x}^3$ – using the inverse of the matrix on the left in (13.80), provided that it exists. In fact, it can be shown – using results for the inverses of partitioned matrices (Appendix A) and letting $\mathbf{R} = \mathbf{G}^{-1}\mathbf{B}$ – that

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} & \mathbf{R}^3\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix} \tag{13.81}$$

For an n-sector economy this will be a square matrix of order $4n$. For a time horizon of T years, this matrix will be of order $(T + 1)n$; that is, it can become fairly large for "reasonable" problems. For a ten-year planning problem in a 100-sector economy this matrix will be 1100×1100.

The particular structure of these equations when $\mathbf{x}^{T+1} = \mathbf{0}$, as in (13.80), allows for a simple recursive solution procedure. Given $\mathbf{f}^3$, find $\mathbf{x}^3$ from

$$\mathbf{x}^3 = \mathbf{G}^{-1}\mathbf{f}^3 \tag{13.82}$$

Using this value for $\mathbf{x}^3$, find $\mathbf{x}^2$ from the third equation in (13.79) as

$$\mathbf{x}^2 = \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^3 + \mathbf{f}^2) = \mathbf{G}^{-1}(\mathbf{B}\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{f}^2) = \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{G}^{-1}\mathbf{f}^2 \tag{13.83}$$

43 Bródy (1995) calls this the "doom" or "doomsday" scenario (meaning, essentially, that the world ends at the end of period T). This reference includes an examination of alternative "truncations" of the matrix in (13.79) and discusses the alternative scenarios that they reflect.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 643.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=677

In similar fashion, knowing $\mathbf{x}^3$ and $\mathbf{x}^2$,

$$\begin{aligned}\mathbf{x}^1 &= \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^2 + \mathbf{f}^1) = \mathbf{G}^{-1}[\mathbf{B}(\mathbf{R}\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{G}^{-1}\mathbf{f}^2) + \mathbf{f}^1] \\ &= \mathbf{R}^2\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^2 + \mathbf{G}^{-1}\mathbf{f}^1 \end{aligned} \tag{13.84}$$

and finally

$$\begin{aligned}\mathbf{x}^0 &= \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^1 + \mathbf{f}^0) = \mathbf{G}^{-1}[\mathbf{B}(\mathbf{R}^2\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^2 + \mathbf{G}^{-1}\mathbf{f}^1) + \mathbf{f}^0] \\ &= \mathbf{R}^3\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{R}^2\mathbf{G}^{-1}\mathbf{f}^2 + \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^1 + \mathbf{G}^{-1}\mathbf{f}^0 \end{aligned} \tag{13.85}$$

This approach moves backward in time, starting at the end ($\mathbf{x}^3$) and finishing at the beginning ($\mathbf{x}^0$).[44] As the reader can see, this sequential solution procedure simply carries out the computations embedded in the upper triangular inverse matrix (zeros below the main diagonal).

Instead of assuming that $\mathbf{x}^{T+1} = \mathbf{0}$ in (13.79), we could have some target value of $\mathbf{x}$ for the first post-terminal year; that is, we could specify that $\mathbf{x}^4 = \bar{\mathbf{x}}^4$. Then the matrix structure in (13.80) would be altered only in that $\mathbf{f}^3$ on the right-hand side would be replaced by $\mathbf{f}^3 + \mathbf{B}\bar{\mathbf{x}}^4$. The solution could still be found using the inverse of the matrix on the left of (13.80), or the recursive solution, as in (13.82)–(13.85), could proceed as before.

Alternatively, one can specify that $\mathbf{x}^{T+1} = \mathbf{H}\mathbf{x}^T$, where $\mathbf{H}$ is a diagonal matrix whose elements are exogenously set growth rates for each of the sectors in the first post-terminal year. In that case, the last equation in (13.79) would be $\mathbf{G}\mathbf{x}^3 - \mathbf{B}\mathbf{H}\mathbf{x}^3 = \mathbf{f}^3$. The matrix structure would be

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{G} - \mathbf{B}\mathbf{H}) \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.86}$$

and solution procedures would be as above.

Initial Conditions Alternatively in assessing future impacts of current events, it is often assumed that the initial ($t = 0$) values of all elements in the system are known and then the usefulness of the model comes from its description of the values to be taken by the variables of interest in subsequent years. From that point of view, we would assume that both $\mathbf{f}^0$ *and* $\mathbf{x}^0$ have given initial values. This reduces the system in (13.79)

[44] A particular special case emerges from (13.85). If we are interested in a τ-year planning period with production to satisfy a constant level of final demand, $\mathbf{f}^*$, each year, an extension of the result in (13.85) leads to $\mathbf{x}^0 = [\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots + \mathbf{R}^\tau]\mathbf{G}^{-1}\mathbf{f}^*$. If, as τ gets large, the power series in brackets converges – as we saw in Chapter 2 for the case of $(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^m)$ – then $\mathbf{x}^0 = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{G}^{-1}\mathbf{f}^*$. Using $\mathbf{N}^{-1}\mathbf{M}^{-1} = (\mathbf{M}\mathbf{N})^{-1}$, and since $\mathbf{R} = \mathbf{G}^{-1}\mathbf{B}$, this is $\mathbf{x}^0 = [\mathbf{G}(\mathbf{I} - \mathbf{R})]^{-1}\mathbf{f}^* = (\mathbf{G} - \mathbf{B})^{-1}\mathbf{f}^*$ and so, with $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$, $\mathbf{x}^0 = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^*$. Finally, as $\tau \to \infty$, $\mathbf{x}^0 = \mathbf{x}^1 = \cdots = \mathbf{x} = \mathbf{x}^*$, so $\mathbf{x}^* = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^*$. This reflects the logical limiting case. When final demand is constant and the time horizon infinite, the output level is constant and there is no need for capital growth.

to $4n$ linear equations in $4n$ variables. Then, given exogenous values for $\mathbf{f}^1, \mathbf{f}^2$, and $\mathbf{f}^3$, we could proceed sequentially from $\mathbf{x}^1$ to $\mathbf{x}^4$. As opposed to the backward sequence in (13.82) through (13.85), this one moves forward in time. From (13.79),

$$\begin{aligned} \mathbf{x}^1 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) \\ \mathbf{x}^2 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^1 - \mathbf{f}^1) \\ \mathbf{x}^3 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^2 - \mathbf{f}^2) \\ \mathbf{x}^4 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^3 - \mathbf{f}^3) \end{aligned} \tag{13.87}$$

This sequential solution procedure depends on the existence of $\mathbf{B}^{-1}$.

The results found sequentially in (13.87) can also be found in matrix form if the system in (13.79) is written as

$$\begin{bmatrix} -\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 - \mathbf{G}\mathbf{x}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.88}$$

This reflects the fact that $\mathbf{x}^0$ is now exogenously determined; it disappears from the top of the $\mathbf{x}$ vector on the left and hence the first column in the coefficient matrix in (13.79) also is removed. Then $\mathbf{x}^1$ through $\mathbf{x}^4$ can be found by premultiplying both sides of (13.88) by the inverse of the coefficient matrix on the left, provided that inverse exists. As before, the matrix on the left-hand side of (13.88) will be nonsingular if and only if the matrix on its main diagonal, here $\mathbf{B}$, is nonsingular. Again, repeated use of the results on inverses of partitioned matrices will demonstrate that (letting $\mathbf{S} = \mathbf{B}^{-1}\mathbf{G}$)

$$\begin{bmatrix} -\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{B}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{S}\mathbf{B}^{-1} & -\mathbf{B}^{-1} & \mathbf{0} & \mathbf{0} \\ -\mathbf{S}^2\mathbf{B}^{-1} & -\mathbf{S}\mathbf{B}^{-1} & -\mathbf{B}^{-1} & \mathbf{0} \\ -\mathbf{S}^3\mathbf{B}^{-1} & -\mathbf{S}^2\mathbf{G}\mathbf{B}^{-1} & -\mathbf{S}\mathbf{B}^{-1} & -\mathbf{B}^{-1} \end{bmatrix} \tag{13.89}$$

As opposed to the inverse matrix in the terminal conditions example, above, this inverse is lower triangular (zeros above the main diagonal), and this feature also suggests a recursive approach to solution that is illustrated by the sequence in (13.87), above.

13.4.3 Numerical Example 1

We illustrate the general workings of the dynamic input–output model using hypothetical figures for a two-sector economy. Let

$$\mathbf{A} = \begin{bmatrix} .1 & .2 \\ .3 & .4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} .05 & .001 \\ .001 & .05 \end{bmatrix}; \text{ then } \mathbf{G} = \begin{bmatrix} .95 & -.199 \\ -.299 & .65 \end{bmatrix}.$$

For simplicity, let $T = 2$.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 645.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=679

Terminal Conditions Suppose that $\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, $\mathbf{f}^1 = \begin{bmatrix} 120 \\ 150 \end{bmatrix}$, and $\mathbf{f}^2 = \begin{bmatrix} 140 \\ 200 \end{bmatrix}$. If we assume that $\mathbf{x}^3 = \mathbf{0}$, then, as in (13.82)–(13.85) – but with $T = 2$ rather than $T = 3$ – we can find the backward sequence $\mathbf{x}^2, \mathbf{x}^1, \mathbf{x}^0$. Here $\mathbf{G}^{-1} = \begin{bmatrix} 1.1649 & 0.3566 \\ 0.5358 & 1.7025 \end{bmatrix}$, so

$$\mathbf{x}^2 = \mathbf{G}^{-1}\mathbf{f}^2 = \begin{bmatrix} 234.41 \\ 415.51 \end{bmatrix} \tag{13.90}$$

Then

$$\mathbf{x}^1 = \mathbf{G}^{-1}(\mathbf{f}^1 + \mathbf{B}\mathbf{x}^2) = \begin{bmatrix} 214.91 \\ 361.94 \end{bmatrix} \tag{13.91}$$

and

$$\mathbf{x}^0 = \mathbf{G}^{-1}(\mathbf{f}^0 + \mathbf{B}\mathbf{x}^1) = \begin{bmatrix} 171.62 \\ 260.96 \end{bmatrix} \tag{13.92}$$

Alternatively, using the full matrix form, as in (13.80), where

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{G}^{-1}\mathbf{B}\mathbf{G}^{-1} & (\mathbf{G}^{-1}\mathbf{B})^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{G}^{-1}\mathbf{B}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} 1.1649 & .3566 & .0784 & .0532 & .0061 & .0061 \\ .5358 & 1.7025 & .0791 & .1560 & .0090 & .0149 \\ 0 & 0 & 1.1649 & .3566 & .0784 & .0532 \\ 0 & 0 & .5358 & 1.7025 & .0791 & .1560 \\ 0 & 0 & 0 & 0 & 1.1649 & .3566 \\ 0 & 0 & 0 & 0 & .5358 & 1.7025 \end{bmatrix} \tag{13.93}$$

we could find, simultaneously, these same values for $\mathbf{x}^0, \mathbf{x}^1$, and $\mathbf{x}^2$.

If, instead of $\mathbf{x}^3 = \mathbf{0}$, we specify $\mathbf{x}^3 = \begin{bmatrix} 250 \\ 450 \end{bmatrix}$ (target values for outputs in the first post-terminal year), then $\mathbf{B}\mathbf{x}^3 = \begin{bmatrix} 12.95 \\ 22.75 \end{bmatrix}$, so that only the equation for $\mathbf{x}^2$ changes

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 646.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=680

slightly from the sequence in (13.90)–(13.92), and

$$\mathbf{x}^2 = \mathbf{G}^{-1}(\mathbf{f}^2 + \mathbf{B}\mathbf{x}^3) = \begin{bmatrix} 257.60 \\ 461.18 \end{bmatrix}$$

$$\mathbf{x}^1 = \mathbf{G}^{-1}(\mathbf{f}^1 + \mathbf{B}\mathbf{x}^2) = \begin{bmatrix} 217.13 \\ 366.52 \end{bmatrix}$$

$$\mathbf{x}^0 = \mathbf{G}^{-1}(\mathbf{f}^0 + \mathbf{B}\mathbf{x}^1) = \begin{bmatrix} 171.83 \\ 261.41 \end{bmatrix}$$

In comparison with the $\mathbf{x}^0$, $\mathbf{x}^1$, and $\mathbf{x}^2$ found above when $\mathbf{x}^3 = \mathbf{0}$, the initial-year outputs are affected very little by this change in post-terminal year conditions. However, $\mathbf{x}^1$ is changed more than $\mathbf{x}^0$ and $\mathbf{x}^2$ more than $\mathbf{x}^1$. In matrix form,

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 + \mathbf{B}\mathbf{x}^3 \end{bmatrix}$$

and, using (13.93), the same values of the gross outputs from both sectors in each period can be found simultaneously.

Using the $\mathbf{x}^3 = \mathbf{0}$ example again, let $\mathbf{f}^0 = \mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Then, from the inverse in (13.93), or from the backward recursive procedure, as above, we can find

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \end{bmatrix} = \begin{bmatrix} 166.53 \\ 249.73 \\ \\ 165.31 \\ 247.34 \\ \\ 152.15 \\ 223.83 \end{bmatrix} \tag{13.94}$$

Recall (footnote 39) that with constant final demands, $\mathbf{f}^*$, as the time period lengthens, the results in each $\mathbf{x}^t$ approach $(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^*$. Here $\mathbf{A} = \begin{bmatrix} .1 & .2 \\ .3 & .4 \end{bmatrix}$, so

$$(\mathbf{I} - \mathbf{A})^{-1} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 1.2500 & .4167 \\ .6250 & 1.8750 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 166.67 \\ 250.00 \end{bmatrix}$$

which is closely approximated by $\mathbf{x}^0$ in (13.94), the outputs in the earliest year. As T gets larger, subsequent values of $\mathbf{x}^t$ will also approach $\begin{bmatrix} 166.67 \\ 250.00 \end{bmatrix}$. (The interested reader can confirm this by letting $T = 3, T = 4$, and so on, using the same $\mathbf{A}$ and $\mathbf{B}$ and constant final demand of 100 for both sectors.)

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 647.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=681

Initial Conditions Taking an alternative point of view, suppose

$$\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}, \ \mathbf{f}^1 = \begin{bmatrix} 120 \\ 150 \end{bmatrix}, \ \mathbf{f}^2 = \begin{bmatrix} 140 \\ 200 \end{bmatrix}$$

as before, but let $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 270 \end{bmatrix}$. Originally, with these final demands and $\mathbf{x}^0$ determined endogenously, we found $\mathbf{x}^0 = \begin{bmatrix} 171.61 \\ 260.96 \end{bmatrix}$, as in (13.92). We now select an $\mathbf{x}^0$ that is larger. Here, using the forward recursive procedure of (13.87), with $\mathbf{B}^{-1} = \begin{bmatrix} 20.008 & -0.4 \\ -0.4 & 20.008 \end{bmatrix}$, we find

$$\mathbf{x}^1 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) = \begin{bmatrix} 336.87 \\ 426.87 \end{bmatrix}$$

and

$$\mathbf{x}^2 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^1 - \mathbf{f}^1) = \begin{bmatrix} 2291.66 \\ 488.91 \end{bmatrix}$$

Essentially the same values are found, using the inverse of $\begin{bmatrix} -\mathbf{B} & \mathbf{0} \\ \mathbf{G} & -\mathbf{B} \end{bmatrix}$, as in (13.88). Here this inverse is

$$\begin{bmatrix} -20.008 & 0.4 & 0 & 0 \\ 0.4 & -20.008 & 0 & 0 \\ -384.395 & 92.522 & -20.008 & 0.4 \\ 132.538 & -264.347 & 0.4 & -20.008 \end{bmatrix}$$

This example illustrates that the dynamic input–output model, at least in the simplified form presented here, is very sensitive to the specification of initial conditions. We return to this point in Numerical Example 2, below.

If we use the same structure as in (13.87) and (13.88), but with $\mathbf{x}^0 = \begin{bmatrix} 171.62 \\ 260.96 \end{bmatrix}$, which is the actual initial output found in (13.92) when $\mathbf{x}^0$ is endogenous, we will generate exactly the values of $\mathbf{x}^1$ and $\mathbf{x}^2$ that were found initially in (13.91) and (13.90). Similarly, if we use $\mathbf{x}^0 = \begin{bmatrix} 166.53 \\ 249.73 \end{bmatrix}$ from (13.94), in conjunction with $\mathbf{f}^0 = \mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, we generate exactly the sequence of outputs already found for that example – $\mathbf{x}^1$ and $\mathbf{x}^2$ in (13.94). For either of these earlier examples, if we take the forward sequential approach but with an initial $\mathbf{x}^0$ that is less than that found with $\mathbf{x}^0$ endogenous (and using the same final demands), we will generate one or more negative gross outputs in years after $t = 0$. The values for $\mathbf{x}^0$ in (13.92) and (13.94) represent what is necessary to satisfy the specified sequences of final demands with an economy whose structure is reflected in the given $\mathbf{A}$ and $\mathbf{B}$ matrices, so any initial output that is less than that $\mathbf{x}^0$

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 648.
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will produce a sequence of additions to capital stock that eventually become inadequate for future production. (Recall that, unlike the static input–output case, in the dynamic model it is assumed that all sectors are producing at full capacity.)

13.4.4 Numerical Example 2

In order to illustrate a particularly sensitive feature of the dynamic input–output model in its forward sequential form (starting from initial conditions), we select an alternative capital coefficients matrix. In this new case, sector 1 is far more important as a supplier of capital goods than is sector 2; here $\mathbf{B} = \begin{bmatrix} .05 & .06 \\ .0004 & .0007 \end{bmatrix}$. Using the same $\mathbf{A}$ matrix as in the preceding example, we find $\mathbf{G} = \begin{bmatrix} .95 & -.14 \\ -.2996 & .6007 \end{bmatrix}$. Note that while $\mathbf{B}$ is quite different from the preceding example, the current $\mathbf{G}$ matrix is close to that in Example 1. This is because $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$, and $\mathbf{A}$ is unchanged in the two examples.

Terminal Conditions We use the same sequence of final demands – namely

$$\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}, \ \mathbf{f}^1 = \begin{bmatrix} 120 \\ 150 \end{bmatrix}, \text{ and } \mathbf{f}^2 = \begin{bmatrix} 140 \\ 200 \end{bmatrix}$$

Again, letting $\mathbf{x}^3 = \mathbf{0}$, we can find $\mathbf{x}^2$, $\mathbf{x}^1$, $\mathbf{x}^0$ sequentially, exactly as in (13.90) through (13.92). Here $\mathbf{G}^{-1} = \begin{bmatrix} 1.1361 & .2648 \\ .5667 & 1.7968 \end{bmatrix}$ (which is not a great deal different from $\mathbf{G}^{-1}$ in the previous example) and

$$\mathbf{x}^2 = \begin{bmatrix} 212.01 \\ 438.70 \end{bmatrix}, \ \mathbf{x}^1 = \begin{bmatrix} 218.10 \\ 359.15 \end{bmatrix}, \ \mathbf{x}^0 = \begin{bmatrix} 177.05 \\ 255.35 \end{bmatrix}$$

These results are different from those in the previous example, as is to be expected, but not by much.

Initial Conditions Using the same $\mathbf{f}^0$, $\mathbf{f}^1$ and $\mathbf{f}^2$ along with $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 270 \end{bmatrix}$ from the previous example illustrates the sensitivity problem. Here, because $\mathbf{B}$ has a row of elements that are smaller than any of the elements in the previous capital coefficients matrix, its inverse can be expected to contain at least some larger elements. And indeed it does; here $\mathbf{B}^{-1} = \begin{bmatrix} 63.636 & -5454.545 \\ -36.364 & 4545.455 \end{bmatrix}$, which is very different from its counterpart in the previous example. Thus

$$\mathbf{x}^1 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) = \mathbf{B}^{-1} \begin{bmatrix} 33.20 \\ 8.26 \end{bmatrix} = \begin{bmatrix} -42942 \\ 36338 \end{bmatrix}$$

and, much worse (the results have been rounded),

$$\mathbf{x}^2 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^1 - \mathbf{f}^1) = \mathbf{B}^{-1} \begin{bmatrix} -45882 \\ 24694 \end{bmatrix} = \begin{bmatrix} -192,000,000 \\ 159,000,000 \end{bmatrix}$$

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Cambridge, , GBR: Cambridge University Press, 2009. p 649.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=683

This illustrates that as the elements in one or more rows of $\mathbf{B}$ become small, $\mathbf{B}^{-1}$ contains very large numbers. Here $|\mathbf{B}| = 0.000011$; if one were working with four-decimal accuracy, one would conclude that $\mathbf{B}$ was singular.

Consider the determination of $\mathbf{x}^1$. Rewriting, $\mathbf{B}\mathbf{x}^1 = \mathbf{G}\mathbf{x}^0 - \mathbf{f}^0$, and with $\mathbf{A}$ and $\mathbf{B}$ (and hence $\mathbf{G}$) given, along with $\mathbf{f}^0$, the choice of $\mathbf{x}^0$ then specifies the right-hand side vector for this set of two linear equations in two unknowns. Denote a specific right-hand side vector as $\mathbf{r}^0$. In the easily visualized two-variable case, we could explore the solution-space geometry of the pair of equations. Here

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\ \mathbf{x}^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix},\ \mathbf{r}^0 = \begin{bmatrix} r_1^0 \\ r_2^0 \end{bmatrix}$$

so

$$\begin{aligned} b_{11}x_1^1 + b_{12}x_2^1 &= r_1^0 \\ b_{21}x_1^1 + b_{22}x_2^1 &= r_2^0 \end{aligned}$$

We leave it to the interested reader to make sketches in solution space. However, it is easy to show that both lines will have positive intercepts on the vertical axis (when $\mathbf{r}^0 > \mathbf{0}$, which by definition it must be) and that both will have negative slopes. Then the conditions for the intersection of the two lines to be in the positive quadrant or on its boundaries (that is, $\mathbf{x}^1 \geq \mathbf{0}$) can be derived. The values of x_1^0 and x_2^0 must be chosen so that r_1^0/r_2^0 lies within the bounds set by b_{11}/b_{21} and b_{12}/b_{22}. The generalization to more sectors and to non-negativity of outputs further in the future – $\mathbf{x}^2, \mathbf{x}^3$, and so on – is beyond this text. The point of the illustration is simply to highlight the kinds of problems that can arise in the dynamic model when one wants to calculate forward from initial conditions, using $\mathbf{B}^{-1}$.

Note that in the first numerical example, $b_{11}/b_{21} = 50$ and $b_{12}/b_{22} = 0.02$. In that example, in fact, $(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) = \mathbf{r}^0 = \begin{bmatrix} 18.77 \\ 21.71 \end{bmatrix}$ so that $r_1^0/r_2^0 = 0.86$, which is indeed within the bounds. In the second example, $b_{11}/b_{21} = 125$ and $b_{12}/b_{22} = 85.7$. For our initial choice of $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 270 \end{bmatrix}$, $r_1^0/r_2^0 = 33.2/8.26 = 4.02$, which is outside the admissible range. A choice of $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 256.8 \end{bmatrix}$, however, would lead to $\mathbf{x}^1 \geq \mathbf{0}$, since $r_1^0/r_2^0 = 105.6$, while an initial $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 256.7 \end{bmatrix}$ generates $r_1^0/r_2^0 = 129.1$, which means that $\mathbf{x}^1$ will not be non-negative. By any reasonable definition, this would appear to be extreme sensitivity to initial values.

13.4.5 "Dynamic" Multipliers

The structure of the inverse in (13.81) suggests the possibility of distributing impacts backward over time. (This is described in Leontief, 1970, and it is also discussed in C. K. Liew, 1977, for a regional model and further elaborated in C. J. Liew, 2000 and

2005.) In these cases, it is usual to designate the current (or "target") period as period 0 and the preceding periods as −1, −2, etc. For example, consider the model in (13.80) and (13.81) in "Δ" form

$$\begin{bmatrix} \Delta\mathbf{x}^{-3} \\ \Delta\mathbf{x}^{-2} \\ \Delta\mathbf{x}^{-1} \\ \Delta\mathbf{x}^{0} \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{RG}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} & \mathbf{R}^3\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{RG}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} & \mathbf{RG}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{f}^{-3} \\ \Delta\mathbf{f}^{-2} \\ \Delta\mathbf{f}^{-1} \\ \Delta\mathbf{f}^{0} \end{bmatrix}$$

Let $\Delta\mathbf{f}^0 \neq \mathbf{0}$, $\Delta\mathbf{f}^{-1} = \Delta\mathbf{f}^{-2} = \Delta\mathbf{f}^{-3} = \mathbf{0}$; then the last column of the inverse on the right is seen to distribute the direct and indirect input requirements backward over time from period 0 in which the deliveries are made to final users. Here, $\Delta\mathbf{x}^{-3} = \mathbf{R}^3\mathbf{G}^{-1}\Delta\mathbf{f}^0$, $\Delta\mathbf{x}^{-2} = \mathbf{R}^2\mathbf{G}^{-1}\Delta\mathbf{f}^0$ and $\Delta\mathbf{x}^{-1} = \mathbf{RG}^{-1}\Delta\mathbf{f}^0$; present demands require both current inputs and adequate capital stock to support production of those inputs, meaning production of capital goods in the preceding period, which in turn depends in part on production two periods back, etc.

Notice that this intertemporal influence is not a result of the fact that production takes time, it is entirely the result of the capital goods component of the model in which production for those goods depends on the *changes* in outputs over time, as reflected in $\mathbf{B}(\mathbf{x}^{t+1} - \mathbf{x}^t)$ in (13.77). Approaches to incorporating *production* lags in an input–output model will be explored below, in section 13.4.6.

13.4.6 Turnpike Growth and Dynamic Models

In Chapter 2 we introduced the notion of a completely closed input–output model as $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ or $\mathbf{Ax} = \mathbf{x}$. Recall that such an input–output model is in fact a homogeneous system of linear equations which has a nontrivial solution (one other than $\mathbf{x} = \mathbf{0}$) if and only if $|\mathbf{I} - \mathbf{A}| = 0$.

The corresponding closed dynamic model is

$$\mathbf{Ax}^t + \mathbf{B}(\mathbf{x}^{t+1} - \mathbf{x}^t) = \mathbf{x}^t \tag{13.95}$$

If we assume for simplicity that we can find an $\mathbf{x}^{t+1}$ and $\mathbf{x}^t$ such that all industries grow at the same rate in the economy, say, at rate λ, then

$$\mathbf{x}^{t+1} = \lambda\mathbf{x}^t \tag{13.96}$$

This rate, λ, is often referred to as a *turnpike growth rate* (all industries are growing or declining on the same path – the "turnpike"), and it is interpreted as a general indicator of the "health" of the economy, that is, $\lambda > 1$ indicates that the economy is expanding, $0 < \lambda < 1$ indicates that the economy is contracting, and $\lambda < 0$ indicates that the economy is unstable, that is, experiencing periods of both decline and growth over time. Since λ is really only a theoretical number, how can it be computed? Substituting

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(13.96) into (13.95), we obtain

$$\begin{aligned}\mathbf{A}\mathbf{x}^t + \mathbf{B}(\lambda\mathbf{x}^t - \mathbf{x}^t) &= \mathbf{x}^t \\ \mathbf{B}\lambda\mathbf{x}^t &= (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t \\ \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t &= \lambda\mathbf{x}^t\end{aligned}$$

or

$$\mathbf{Q}\mathbf{x}^t = \lambda\mathbf{x}^t \tag{13.97}$$

where $\mathbf{Q} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B})$. Note that (13.97) has the very interesting feature that a *scalar*, λ, multiplied by $\mathbf{x}^t$, yields precisely the same value as a matrix, $\mathbf{Q}$, postmultiplied by $\mathbf{x}^t$.

Such a problem is well known in applied mathematics as an eigenvalue problem where λ is the *eigenvalue* (sometimes called a characteristic value or latent root), and $\mathbf{x}^t$ corresponding to λ in (13.97), is the *eigenvector* (sometimes called characteristic vector or latent vector). This problem is closely related to the solution of systems of homogeneous linear equations. Note that we can rewrite (13.97) as

$$(\mathbf{Q} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{13.98}$$

for which there is a nontrivial solution if and only if

$$|\mathbf{Q} - \lambda\mathbf{I}| = 0 \tag{13.99}$$

We consider the 2×2 case, with $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ so that

$$|\mathbf{Q} - \lambda\mathbf{I}| = \left|\begin{bmatrix} q_{11} - \lambda & q_{12} \\ q_{21} & q_{22} - \lambda \end{bmatrix}\right| = (q_{11} - \lambda)(q_{22} - \lambda) - q_{12}q_{21} = 0 = \lambda^2 + b\lambda + c$$

where $b = -(q_{11} + q_{22})$ and $c = q_{11}q_{22} - q_{12}q_{21}$. We find the solution to $\mathbf{Q}\mathbf{x} = \lambda\mathbf{x}$ by solving $|\mathbf{Q} - \lambda\mathbf{I}| = 0$ or $\lambda^2 + b\lambda + c = 0$. This is a polynomial (sometimes called the characteristic polynomial) which, when set equal to zero, is called the characteristic equation; in this case it has two solutions, given by

$$\lambda = \frac{-(q_{11} + q_{22}) \pm [(q_{11} + q_{22})^2 - 4(q_{11}q_{22} - q_{12}q_{21})]^{1/2}}{2}$$

Denote these solutions as λ_1 and λ_2. The turnpike growth rate is defined to be the largest eigenvalue found (see Carter, 1974), which we define as $\lambda_{\max}$.

Example

$$\text{Suppose } \mathbf{Q} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B}) = \begin{bmatrix} 1.0 & .5 \\ 2.0 & 1.0 \end{bmatrix}, \text{ then}$$

$$|\mathbf{Q} - \lambda\mathbf{I}| = (1-\lambda)(1-\lambda) - 1 = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

so $\lambda_1 = 0$ and $\lambda_2 = 2$. The turnpike growth rate is $\lambda_{max} = \lambda_2 = 2$. As mentioned earlier, if $\lambda_{max} < 0$ then the economy is unstable, that is, oscillating. The interpretation of negative λ's can be specified more precisely by relating it to the solution of a system of ordinary differential equations, but this is beyond the scope of this text. Carter (1974) and Leontief and Duchin (1986) examine the notion of turnpike growth as an indicator economic stability resulting from changes in technology in the United States.

13.4.7 Alternative Input–Output Dynamics

In the standard input–output model, $\mathbf{x} = \mathbf{Lf}$, there is no consideration of the fact that production takes time; results are independent of time in the sense that $\mathbf{f}^{new}$ leads to $\mathbf{x}^{new}$ (via $\mathbf{x}^{new} = \mathbf{Lf}^{new}$). This is generally interpreted in something like the following fashion: "new demands, $\mathbf{f}^{new}$, next period will lead to new outputs, $\mathbf{x}^{new}$, next period," ignoring that sectors will generally have production lags (of differing length for different sectors). This missing temporal characteristic of the input–output model was noted in the early work of Dorfman, Samuelson and Solow (1958, pp. 253–254) where the authors comment on the absence of a "time" aspect to the round-by-round process of the power series for the Leontief inverse.

> Needless to say, the rounds of which we speak do not take place in calendar time, with the second round following the first . . . Artificial computational time is involved, and if we insist on giving a calendar-time interpretation we must think of the . . . process as showing how much production must be started many periods back if we are to meet the new consumption targets today.

However, as observed by Mules (1983, p. 197), with respect to the assumptions implicit in using input–output multipliers,

> The traditional multiplier does not stipulate the time taken to realize effects, assuming instead that they usually occur almost immediately or within the space of one year (a year being the usual accounting period for which input–output data is compiled).

Starting around the mid-1980s, research emerged on ways to incorporate the notion of time lags in production in an input–output framework. [References include Mules, 1983; ten Raa, 1986, 2005 (Chapter 13); Romanoff and Levine, 1986, 1990; and Cole, 1988, 1997, 1999b.] Mules (p. 199) makes the assumption that each round of the power series process does in fact take a finite period of (calendar) time, suggesting that a typical period may be a month or a quarter. He further assumes that each sector is able to respond in each period to the demands made upon it in the previous period, but with varying lags in this production response. As an illustration, he suggests a five-period lag for primary sectors, one period for manufacturing and no lag (that is, delivery next period) for services. Simulation exercises lead to the conclusion that "… on some occasions there may be a significant proportion of multiplier effects still outstanding after one year has elapsed. We may be in error if we assume that all effects have occurred near the time of the original stimulus" (Mules, 1983, p. 204).

This problem was also addressed in the work of Romanoff and Levine. A good deal of their work on what they call the sequential interindustry model (SIM) appears in unpublished discussion papers from the Regional Science Research Center (initially in Cambridge, Mass. and later in Lexington, Mass.). Possibly the first is dated 1980, so it precedes (and is cited by) Mules. The authors recognize the fact that "… it takes time for each industry to produce its product beforehand and supply its own final demand and that of the directly demanding industries, for the latter to use as inputs to their own production" (Romanoff and Levine, 1990, pp. 1–2). A given a_{ij} is modeled as distributed backward (over discrete time intervals); $a_{ij}(k)(k = 0, -1, -2, \ldots)$ is the fraction of a_{ij} (per-unit input of i by j) that occurs k periods before completion of production by j.

Ten Raa (1986) and Cole (1988) identify technical coefficients as non-negative (continuous) distributions along the negative time axis (that is, backwards in time from "now"). As in the dynamic models in sections 13.4.1–13.4.5, ten Raa also considers capital accumulation. The specific nature, characteristics, and properties of the assumed distributions are beyond the level of this text. The interested reader is referred to the cited literature and additional references in those articles. Cole has successfully applied his distributed-lag framework in a number of studies, especially at the small-area level. In Cole (1989) the illustration is a plant closure in Western New York. Assumed lags are: 3 months for production sectors, 4 months for households, 18 months for local government activities and 36 months for investments.[45] In Cole (1999a) there is a stylized illustration, including Miyazawa interrelational multiplier aspects, for a community (an inner-city neighborhood in Buffalo, New York).[46]

13.5 Summary

In this chapter we have explored several applications and variations of the input–output framework. Structural decomposition analysis presents an approach to disentangling the sources of change in some aspect of an economy into its component parts – for example relating output changes to changes in demand and technology. We saw that it is further possible to decompose the demand and technology changes into further underlying components. And further layers of decomposition are also possible. In a large (many-sector) input–output model this approach rapidly generates a very large set of results which are generally difficult to interpret without some kind of aggregation

[45] In comparing his approach with that of ten Raa or Romanoff and Levine, Cole (1989, p. 106) suggests that the required computations needed in either of those approaches "… are still complex in any practical situation."

[46] A vigorous exchange in print – Jackson, Madden and Bowman (1997) → Cole (1997) → Jackson and Madden (1999) → Cole (1999b) – provides several illustrations of Cole's approach and comparisons of Cole's work with that of ten Raa and of Romanoff and Levine.

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(for example, finding averages); this, as usual, removes much of the detail which the input–output model provides.

We also explored the variations that arise when the model is used to assess the impact of exogenously specified outputs for one or more sectors (rather than final demands), or when a new sector is introduced into the economy. Finally, we sketched the basic features of the dynamic version of an input-model, where production for current input use is coupled with production for capital goods. The dynamic model has been much less widely embraced in real-world applications, although there have been notable exceptions – including the work of Miernyk and his associates at the regional level in the 1970s, Almon (1970, and other publications associated with the long-running INFORUM project at the University of Maryland) and Leontief and Duchin as well as Duchin and her associates (for example, Leontief and Duchin, 1986; Duchin and Szyld, 1985). An alternative approach to dynamics is represented in the sequential input–output model and its variants that include the recognition of production lags in an economy.

Appendix 13.1 Alternative Decompositions of $\mathbf{x} = \mathbf{LBf}$

Alternative views of an input–output equation like $\mathbf{x} = \mathbf{LBf}$ will generate somewhat different decompositions. We explore three variations in this Appendix.

1. Using (13.10) directly on $\mathbf{x} = \mathbf{LBf}$ gives

$$\Delta\mathbf{x} = \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{L}} + \underbrace{(1/2)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]}_{\text{Effect of }\Delta\mathbf{B}} + \underbrace{(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of }\Delta\mathbf{f}}$$

2. If we combine $\mathbf{L}$ and $\mathbf{B}$, so that $\mathbf{M} = \mathbf{LB}$ and $\mathbf{x} = \mathbf{Mf}$, and then use (13.7),

$$\Delta\mathbf{x} = (1/2)(\Delta\mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f})$$

Since $\mathbf{M} = \mathbf{LB}$,

$$\Delta\mathbf{M} = (1/2)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})$$

so that

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)[(1/2)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})](\mathbf{f}^0 + \mathbf{f}^1) \\ &\quad + (1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f}) \\ &= \underbrace{(1/4)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1)(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{L}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{B}} \\ &\quad + \underbrace{(1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f})}_{\text{Effect of }\Delta\mathbf{f}}\end{aligned}$$

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And, again since $\mathbf{M} = \mathbf{LB}$, the last term is $\underbrace{(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of } \Delta\mathbf{f}}$, as in (1).

3. If we combine $\mathbf{B}$ and $\mathbf{f}$, so that $\mathbf{y} = \mathbf{Bf}$ and $\mathbf{x} = \mathbf{Ly}$, and then use (13.7),

$$\Delta\mathbf{x} = (1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{y})$$

Since $\mathbf{y} = \mathbf{Bf}$,

$$\Delta\mathbf{y} = (1/2)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})$$

so that

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)[(1/2)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) \\ &\quad + (1/2)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})] \\ &= \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1)}_{\text{Effect of } \Delta\mathbf{L}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{B}} \\ &\quad + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of } \Delta\mathbf{f}}\end{aligned}$$

and since $\mathbf{y} = \mathbf{Bf}$, the first term is $\underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{L}}$, as in (1).

Table A13.1.1 summarizes these results. Terms that do not appear in Equation (1) are boxed. For example, in Equation (2), $\Delta\mathbf{L}$ appears in two terms – $(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ and $(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1 + \mathbf{B}^1\mathbf{f}^0)$ – but each is weighted by (1/4) instead of the (1/2) in Equation (1). The amount by which $(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ differs from $(1/4)[(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)+(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1 + \mathbf{B}^1\mathbf{f}^0)]$ depends entirely on the difference between $(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)$ and $(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)$. Similar observations can be made for the weightings on $\Delta\mathbf{B}$ in Equations (2) and (3) vs. Equation (1) and on the weighting on $\Delta\mathbf{f}$ in Equation (3) vs. Equations (1) and (2).

Appendix 13.2 Exogenous Specification of Some Elements of x

A13.2.1 The General Case: An n-sector Model with k Endogenous Outputs

The general representation for an n-sector model with (the first) k gross outputs and (the last) $(n-k)$ final demands endogenous was given in (13.57) in the text as

$$\begin{bmatrix}(\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\underset{(k\times 1)}{\mathbf{x}^{en}} \\ \underset{[(n-k)\times 1]}{\mathbf{f}^{en}}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22})\end{bmatrix}\begin{bmatrix}\underset{(k\times 1)}{\mathbf{f}^{ex}} \\ \underset{[(n-k)\times 1]}{\mathbf{x}^{ex}}\end{bmatrix} \quad \text{(A13.2.1)}$$

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Table A13.1.1 Alternative Decompositions of $\mathbf{x} = \mathbf{LBf}$

Alternative	Effect of $\Delta\mathbf{L}$	Effect of $\Delta\mathbf{B}$	Effect of $\Delta\mathbf{f}$
(1)	$(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)$	$(1/2)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]$	$(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})$
(2)	$(1/4)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1)(\mathbf{f}^0 + \mathbf{f}^1) =$ $(1/4)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1) +$ $\boxed{(1/4)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1 + \mathbf{B}^1\mathbf{f}^0)}$	$(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) =$ $(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0] +$ $\boxed{(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^1]}$	$(1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f}) =$ $(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})$
(3)	$(1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) =$ $(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)$	$(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) =$ $(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0] +$ $\boxed{(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^1]}$	$(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f}) =$ $(1/4)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f}) +$ $\boxed{(1/4)(\mathbf{L}^0\mathbf{B}^1 + \mathbf{L}^1\mathbf{B}^0)(\Delta\mathbf{f})}$

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Letting $\mathbf{M} = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix}$, and using results from Appendix A on inverses of partitioned matrices,

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & -\mathbf{I} \end{bmatrix}$$

[where $(\mathbf{I}-\mathbf{A}^{(k)})^{-1} = \mathbf{L}^{(k)}$] and so

$$\mathbf{M}^{-1}\mathbf{N} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & (\mathbf{I}-\mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12} \end{bmatrix}$$

This product reflects not only the results on inverses of partitioned matrices but also the specific structure of those matrices in (A13.2.1) – especially the locations of **0** and **I** submatrices and their influence in the partitioned matrix multiplication. Thus

$$\begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & (\mathbf{I}-\mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix} \tag{A13.2.2}$$

[This is (13.58) in the text.]

If $\mathbf{f}^{ex} = \mathbf{0}$, the influence of the specified exogenous outputs, $\mathbf{x}^{ex}$, on the endogenous outputs, $\mathbf{x}^{en}$, is given by

$$\mathbf{x}^{en} = \mathbf{L}^{(k)}\mathbf{A}_{12}\mathbf{x}^{ex} \tag{A13.2.3}$$

which was (in "Δ" form) (13.59) in the text.

A13.2.2 The Output-to-Output Multiplier Matrix

For a three-sector model, we saw that an "output–to–output" multiplier matrix was created from $\mathbf{L}^{(3)}$ through division of each element in a column by the on-diagonal element for that column, namely

$$\mathbf{L}^{(3)} = \begin{bmatrix} l_{11}^{(3)} & l_{12}^{(3)} & l_{13}^{(3)} \\ l_{21}^{(3)} & l_{22}^{(3)} & l_{23}^{(3)} \\ l_{31}^{(3)} & l_{32}^{(3)} & l_{33}^{(3)} \end{bmatrix} \text{ and } \mathbf{L}^{(3)*} = \begin{bmatrix} \frac{l_{11}^{(3)}}{l_{11}^{(3)}} & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & \frac{l_{22}^{(3)}}{l_{22}^{(3)}} & \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & \frac{l_{33}^{(3)}}{l_{33}^{(3)}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & 1 & \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & 1 \end{bmatrix}$$

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A13.2.3 The Inverse of a Partitioned $(\mathbf{I}-\mathbf{A}^{(n)})$ *Matrix*

Let

$$(\mathbf{I}-\mathbf{A}^{(n)}) = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & (\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix} = \begin{bmatrix} \underset{(k\times k)}{\mathbf{E}} & \underset{[k\times(n-k)]}{\mathbf{F}} \\ \underset{[(n-k)\times k]}{\mathbf{G}} & \underset{[(n-k)\times(n-k)]}{\mathbf{H}} \end{bmatrix} \quad \text{(A13.2.4)}$$

Then, again using results on inverses of partitioned matrices,

$$(\mathbf{I}-\mathbf{A}^{(n)})^{-1} = \mathbf{L}^{(n)} = \begin{bmatrix} \underset{(k\times k)}{\mathbf{S}} & \underset{[k\times(n-k)]}{\mathbf{T}} \\ \underset{[(n-k)\times k]}{\mathbf{U}} & \underset{[(n-k)\times(n-k)]}{\mathbf{V}} \end{bmatrix} \quad \text{(A13.2.5)}$$

The important result from Appendix A is that $\mathbf{T} = -\mathbf{E}^{-1}\mathbf{F}\mathbf{V}$, or

$$-\mathbf{E}^{-1}\mathbf{F} = \mathbf{T}\mathbf{V}^{-1} \quad \text{(A13.2.6)}$$

A13.2.4 The Case of $k = 2, n = 3$

We now use the results in the preceding sections of this Appendix to examine the specific case of a three-sector model with $x_3 = \bar{x}_3$. This was the subject matter of the examples in section 13.2.3 in the text. In this case, (A13.2.3) becomes

$$\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)}\mathbf{A}_{12}\bar{x}_3 = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3 \quad \text{(A13.2.7)}$$

In terms of (A13.2.4), $\mathbf{E} = (\mathbf{I}-\mathbf{A}^{(2)})$ and $\mathbf{F} = \begin{bmatrix} -a_{13} \\ -a_{23} \end{bmatrix}$, so (A13.2.7) can be written

$$\mathbf{x}^{en} = -\mathbf{E}^{-1}\mathbf{F}\bar{x}_3 \quad \text{(A13.2.8)}$$

In the alternative approach, the exogenous specification of $x_3 = \bar{x}_3$ is represented by the 3×1 vector $\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ \bar{x}_3 \end{bmatrix}$ and

$$\underset{(3\times 1)}{\mathbf{x}^*} = \underset{(3\times 3)}{\mathbf{L}^{(3)*}} \underset{(3\times 1)}{\bar{\mathbf{x}}} \quad \text{(A13.2.9)}$$

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where $\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Here

$$\mathbf{L}^{(3)*} = \begin{bmatrix} 1 & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & 1 & \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11}^{(3)*} & \mathbf{L}_{12}^{(3)*} \\ \mathbf{L}_{21}^{(3)*} & \mathbf{L}_{22}^{(3)*} \end{bmatrix}$$

and (A13.2.9) can be alternatively expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11}^{(3)*} & \mathbf{L}_{12}^{(3)*} \\ \mathbf{L}_{21}^{(3)*} & \mathbf{L}_{22}^{(3)*} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \bar{x}_3 \end{bmatrix}$$

In particular,

$$\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}_{12}^{(3)*}\bar{x}_3 \qquad \text{(A13.2.10)}$$

and

$$x_3 = \mathbf{L}_{22}^{(3)*}\bar{x}_3 = \bar{x}_3$$

From (A13.2.5), $\mathbf{T} = \begin{bmatrix} l_{13}^{(3)} \\ l_{23}^{(3)} \end{bmatrix}$, $\mathbf{V} = [l_{33}^{(3)}]$ and so $\mathbf{TV}^{-1} = \begin{bmatrix} \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \end{bmatrix} = \begin{bmatrix} l_{13}^{(3)*} \\ l_{23}^{(3)*} \end{bmatrix}$. Therefore, the results in (A13.2.10) can be expressed as

$$\mathbf{x}^{en} = \begin{bmatrix} l_{13}^{(3)*} \\ l_{23}^{(3)*} \end{bmatrix} \bar{x}_3 = \mathbf{TV}^{-1}\bar{x}_3 \qquad \text{(A13.2.11)}$$

Conclusion: since $-\mathbf{E}^{-1}\mathbf{F} = \mathbf{TV}^{-1}$ (A13.2.6), the results in (A13.2.8) and (A13.2.11) are equivalent. This will be true for an input–output model of any size in which x_n is made endogenous. It will *not* be true in an input–output model in which more than one output is made exogenous. We examine why in the next section.

A13.2.5 The Case of $k = 1$, $n = 3$

The case in which more than one output is exogenous can be illustrated for a three-sector model in which $x_2 = \bar{x}_2$ and $x_3 = \bar{x}_3$. The results generalize to any n with $k < (n - 1)$.

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For this example, where $(n-k)=2$,

$$\mathbf{M}=\begin{bmatrix}(1-a_{11}) & 0 & 0\\ -a_{21} & -1 & 0\\ -a_{31} & 0 & -1\end{bmatrix},\quad \mathbf{N}=\begin{bmatrix}1 & a_{12} & a_{13}\\ 0 & -(1-a_{22}) & a_{23}\\ 0 & a_{32} & -(1-a_{33})\end{bmatrix},$$

$$\text{and}\quad \mathbf{M}^{-1}=\begin{bmatrix}l_{11}^{(1)} & 0 & 0\\ -a_{21}l_{11}^{(1)} & -1 & 0\\ -a_{31}l_{11}^{(1)} & 0 & -1\end{bmatrix}$$

The parallel to (A13.2.7) is

$$\mathbf{x}^{en}=[x_1]=\mathbf{L}^{(1)}\mathbf{A}_{12}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=(1-a_{11})^{-1}\begin{bmatrix}a_{12} & a_{13}\end{bmatrix}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

From section A13.2.3, it is easily established that $\mathbf{E}^{-1}=(1-a_{11})^{-1}$ and $\mathbf{F}=\begin{bmatrix}-a_{12} & -a_{13}\end{bmatrix}$ and so

$$[x_1]=-\mathbf{E}^{-1}\mathbf{F}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

Here we find

$$\mathbf{T}=\begin{bmatrix}l_{12}^{(3)} & l_{13}^{(3)}\end{bmatrix},\quad \mathbf{V}=\begin{bmatrix}l_{22}^{(3)} & l_{23}^{(3)}\\ l_{32}^{(3)} & l_{33}^{(3)}\end{bmatrix}\ \text{and}\ \ \mathbf{TV}^{-1}=\begin{bmatrix}l_{12}^{(3)} & l_{13}^{(3)}\end{bmatrix}\begin{bmatrix}l_{22}^{(3)} & l_{23}^{(3)}\\ l_{32}^{(3)} & l_{33}^{(3)}\end{bmatrix}^{-1}$$

Notice how the dimensions of $\mathbf{T}$ and $\mathbf{V}$ have been altered in this case. Because of (A13.2.6), we can write

$$[x_1]=\mathbf{TV}^{-1}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=\begin{bmatrix}l_{12}^{(3)} & l_{13}^{(3)}\end{bmatrix}\begin{bmatrix}l_{22}^{(3)} & l_{23}^{(3)}\\ l_{32}^{(3)} & l_{33}^{(3)}\end{bmatrix}^{-1}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}\tag{A13.2.12}$$

On the other hand, now

$$\mathbf{L}^{(3)*}=\begin{bmatrix}\frac{l_{11}^{(3)}}{l_{11}^{(3)}} & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & \frac{l_{22}^{(3)}}{l_{22}^{(3)}} & \frac{l_{23}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & \frac{l_{33}^{(3)}}{l_{33}^{(3)}}\end{bmatrix}=\begin{bmatrix}1 & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & 1 & \frac{l_{23}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & 1\end{bmatrix}=\begin{bmatrix}\mathbf{L}_{11}^{(3)*} & \mathbf{L}_{12}^{(3)*}\\ \mathbf{L}_{21}^{(3)*} & \mathbf{L}_{22}^{(3)*}\end{bmatrix}$$

(notice how the matrix partitions have moved), and the parallel to the result in (A13.2.10) becomes

$$\mathbf{x}^{en}=[x_1]=\mathbf{L}_{12}^{(3)*}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=\begin{bmatrix}l_{12}^{(3)*} & l_{13}^{(3)*}\end{bmatrix}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=\begin{bmatrix}\frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}}\end{bmatrix}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}\tag{A13.2.13}$$

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Clearly, the results in (A13.2.12) differ from those in (A13.2.13). Only the results in (A13.2.12) are valid, because they are derived from the fundamental input–output model for mixed exogenous/endogenous variables – (13.57) in the text.

The problem occurs because when more than one output is made exogenous – $k < (n-1)$ [or $(n-k) > 1$] – the immediate consequence is that $\mathbf{T}$ changes from a column vector to a matrix (with $n-k$ columns) and $\mathbf{V}^{-1}$ changes from the reciprocal of a scalar to the inverse of an $(n-k) \times (n-k)$ matrix. As a result, the operation $\mathbf{TV}^{-1}$ no longer produces a column of elements that have been divided by the on-diagonal element in that column but rather a matrix of elements that differ from the elements in $\mathbf{L}^*$. (Notice that if $\mathbf{V}$ were a *diagonal* matrix then the operation $\mathbf{TV}^{-1}$ would in fact produce a matrix with elements from $\mathbf{L}^*$; but since $\mathbf{V}$ is a submatrix from $\mathbf{L}^{(n)}$ it will *not* be diagonal.)

A13.2.6 "Extracting" the Last (n − k) Sectors

Assume, again, that outputs for the last $(n-k)$ sectors in an n-sector input–output model have been made exogenous. Then modify the $\mathbf{A}^{(n)}$ coefficient matrix by replacing all coefficients in the last $(n-k)$ rows with zeros, creating $\tilde{\mathbf{A}}^{(n)} = \begin{bmatrix} \underset{(k\times k)}{\mathbf{A}_{11}} & \underset{[k\times(n-k)]}{\mathbf{A}_{12}} \\ \underset{[(n-k)\times k]}{\mathbf{0}} & \underset{[(n-k)\times(n-k)]}{\mathbf{0}} \end{bmatrix}$ with an associated

$$(\mathbf{I} - \tilde{\mathbf{A}}^{(n)}) = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{(k)}) & -\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} \text{ and}$$

$$(\mathbf{I} - \tilde{\mathbf{A}}^{(n)})^{-1} = \tilde{\mathbf{L}}^{(n)} = \begin{bmatrix} \mathbf{S} & \mathbf{T} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}$$

Using results from Appendix A, $\mathbf{G} = \mathbf{0}$ means that $\mathbf{U} = \mathbf{0}$ and $\mathbf{S} = (\mathbf{I} - \mathbf{A}^{(k)})^{-1} = \mathbf{L}^{(k)}$. Also, since $\mathbf{H} = \mathbf{I}$, $\mathbf{V} = \mathbf{I}$. Finally, $\mathbf{T} = \mathbf{L}^{(k)}\mathbf{A}_{12}$ and so

$$\tilde{\mathbf{L}}^{(n)} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Finally, then,

$$\begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} = \tilde{\mathbf{L}}^{(n)} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix}$$

and the results for $\mathbf{x}^{en}$ will be exactly the same as given in (A13.2.2) since the upper two submatrices here are identical to those in that equation. [We leave it as an exercise for the interested reader to show that it is immaterial whether the sectors with exogenous outputs are represented as the *last* $(n-k)$ sectors, as here, or as the *first* $(n-k)$ sectors.]

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Problems

13.1 Consider two input–output economies specified by

$$\mathbf{Z}^0 = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 2 & 25 \\ 20 & 40 & 60 \end{bmatrix}, \quad \mathbf{f}^0 = \begin{bmatrix} 60 \\ 40 \\ 55 \end{bmatrix}, \quad \mathbf{Z}^1 = \begin{bmatrix} 15 & 25 & 40 \\ 12 & 7.5 & 30 \\ 10 & 30 & 40 \end{bmatrix}, \quad \mathbf{f}^1 = \begin{bmatrix} 75 \\ 55 \\ 40 \end{bmatrix}$$

We seek to measure how the economy has changed in structure in one year, specified by $\mathbf{Z}^1$ and $\mathbf{f}^1$, relative to an earlier year for the same economy, specified by $\mathbf{Z}^0$ and $\mathbf{f}^0$. Compute for each sector the change in total output between the two years that was attributable to changing final demand or to changing technology.

13.2 Consider an input–output economy specified by $\mathbf{Z} = \begin{bmatrix} 14 & 76 & 46 \\ 54 & 22 & 5 \\ 68 & 71 & 94 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix}$ where the three industrial sectors are manufacturing, oil, and electricity.

a. Suppose economic forecasts determine that total domestic output for oil and electricity will remain unchanged in the next year and final demand for manufactured goods will increase by 30 percent. What would be the input–output projections of final demand for oil and electricity and the total output of manufacturing?
b. If instead the final demand for manufactured goods increased by 50 percent instead of 30 percent, what are the new projections of final demand for oil and electricity and the total output of manufacturing?

13.3 Consider the impact on the economy of Problem 2.1 of the establishment of a new economic sector, finance, and insurance (sector 3).

a. Suppose you know that the total output of this new sector will be \$900 during the current year (its first year of operation), and that its needs for agricultural and manufactured goods are represented by $a_{13} = 0.001$ and $a_{23} = 0.07$. In the absence of any further information, what would you estimate to be the impact of this new sector on the economy?
b. You later learn (1) that the agriculture and manufacturing sectors bought \$20 and \$40 in finance and insurance services last year from foreign firms (i.e., that they imported these inputs), and (2) that sector 3 will use \$15 of its own product for each \$100 worth of its output. Assuming that they will now buy from the domestic sector, how might you now assess the impact of the new sector on this economy?

13.4 Recall the Czaria economy from problem 12.1. Next year's projected total outputs in millions of dollars for agriculture, mining, and civilian manufacturing in Czaria are 4,558, 5,665, and 5,079, respectively, and final demand of military manufactured products is projected to be \$2,050 million. Compute the GDP and total gross production of the economy next year.

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13.5 Consider an input–output economy with technical coefficients defined as $\mathbf{A} = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$ and capital coefficients defined as $\mathbf{B} = \begin{bmatrix} .01 & .003 \\ .005 & .020 \end{bmatrix}$. Current final demand is $\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ and the projections for the next three years for final demand are given by $\mathbf{f}^1 = \begin{bmatrix} 125 \\ 160 \end{bmatrix}$, $\mathbf{f}^2 = \begin{bmatrix} 150 \\ 175 \end{bmatrix}$ and $\mathbf{f}^3 = \begin{bmatrix} 185 \\ 200 \end{bmatrix}$. We are not interested in total output for beyond the projection three years, but what would be the projections of total output for this economy in the next three years?

13.6 Consider the following closed dynamic input–output model, $\mathbf{Ax} + \mathbf{B}(\mathbf{x}' - \mathbf{x}) = \mathbf{x}$ where $\mathbf{x}'$ = future outputs, $\mathbf{x}$ = current outputs, and where $\mathbf{A} = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$. Assume that $\mathbf{x}' = \lambda \mathbf{x}$, where λ is some scalar (the turnpike growth rate); compute λ.

13.7 Given the closed dynamic input–output model $\mathbf{Ax} + \mathbf{B}(\mathbf{x}' - \mathbf{x}) = \mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

a. Compute the turnpike growth rate for this example.
b. If both the capital coefficients for the first industry (the first column of $\mathbf{B}$) are changed to 0.1, then what is the new turnpike growth rate and what has happened to the apparent "health" of the economy?

13.8 Consider an input–output economy with technical coefficients defined as $\mathbf{A} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.5 \end{bmatrix}$ and capital coefficients defined as $\mathbf{B} = \begin{bmatrix} .02 & .002 \\ .003 & .01 \end{bmatrix}$. Current final demand is $\mathbf{f}^0 = \begin{bmatrix} 185 \\ 200 \end{bmatrix}$ and final demands for the previous three years are given by $\mathbf{f}^{-1} = \begin{bmatrix} 150 \\ 175 \end{bmatrix}$, $\mathbf{f}^{-2} = \begin{bmatrix} 125 \\ 160 \end{bmatrix}$, and $\mathbf{f}^{-3} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Compute the "dynamic" multipliers for this economy that show how direct and indirect input requirements for final demands in period 0 are distributed backward over time for the previous three years.

13.9 Consider **A**, **L**, and **f** for the US economy provided in Appendix B for the years 1972 and 2002. Compute the changes in total output between 1972 and 2002 for all sectors attributed to changes in final demand and to changes in technology.

13.10 Consider the 2005 US input–output table provided in Appendix B. Suppose our economic forecast projects for 2010 a 10 percent growth in final demand for agriculture, mining, and construction, a 5 percent growth in final demand for manufactured goods, and a 6 percent growth in total output for the trade, transportation, utilities, services, and other economic sectors. What are the corresponding input–output estimates of total output for agriculture, mining, construction, and manufacturing as well

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as the estimates of final demand for trade, transportation, utilities, services, and other economic sectors?

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