Appendix A Matrix Algebra for Input–Output Models

A.1 Introduction

A matrix is a collection of elements arranged in a grid – a pattern of rows and columns. In all cases that will be of interest to the topics in this book, the elements will be numbers whose values either are known or are unknown and to be determined. Matrices are defined in this "rectangular" way so that they can be used to represent systems of linear relations among variables, which is exactly the structure of an input–output model.

The general case, then, will be a matrix with m rows and n columns. If $m = 2$ and $n = 3$, and using double subscript notation, a_{ij}, to denote the element in row i and column j of the matrix, we have

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

A particular example of such a matrix might be

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix}$$

These are said to be 2×3 (read "2 by 3") matrices or matrices of *dimension* 2 by 3. Dimensions are often denoted in parentheses underneath the matrix, as in $\underset{(2\times 3)}{\mathbf{M}}$.

When $m = n$ the matrix is *square*; in this case it is often referred to as a matrix *of order m* (or *of order n*, since they are the same). If $m = 1$ (a matrix with only one row) it is called a *row vector*; if $n = 1$ (a matrix with only one column) it is called a *column vector*.[1] We adhere to the convention of using upper-case bold letters for matrices, lower-case bold letters for vectors, and italicized letters for elements of matrices and vectors. (In matrix algebra, an ordinary number is called a *scalar*.)

[1] The ultimate in shrinkage is when $m = n = 1$, a matrix with only one element. These will not be needed for input–output models.

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Cambridge, , GBR: Cambridge University Press, 2009. p 688.
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A.2 Matrix Operations: Addition and Subtraction

A.2.1 Addition

Addition of matrices, say $\mathbf{A} + \mathbf{B}$, is accomplished by the simple rule of adding elements *in corresponding positions*. This means $a_{ij} + b_{ij}$ for all i and j; and this, in turn, means that only matrices that have exactly the same dimensions can be added. Given $\mathbf{M}$, above, and $\underset{(2\times3)}{\mathbf{N}} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, their sum, $\mathbf{S} = \mathbf{M} + \mathbf{N}$, will be

$$\underset{(2\times3)}{\mathbf{S}} = \begin{bmatrix} 3 & 3 & 6 \\ 7 & 8 & 13 \end{bmatrix}$$

A.2.2 Subtraction

Subtraction is defined in a completely parallel way, namely subtraction of elements *in corresponding positions*. So, again, only matrices of exactly the same dimensions can be subtracted. For example, $\mathbf{D} = \mathbf{M} - \mathbf{N}$ will be

$$\underset{(2\times3)}{\mathbf{D}} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 4 & 11 \end{bmatrix}$$

A.2.3 Equality

The notion of equality of two (or more) matrices is also very straightforward. Two matrices are equal if they have the same dimensions and if the elements in corresponding positions are equal. So $\mathbf{A} = \mathbf{B}$ when $a_{ij} = b_{ij}$, for all i and j.

A.2.4 The Null Matrix

A zero in ordinary algebra is the number which, when added to (or subtracted from) another number leaves that number unchanged. The completely parallel notion in matrix algebra is a *null matrix*, simply defined as a matrix containing only zeros. Define $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; then it is obvious that $\mathbf{M} + \mathbf{0} = \mathbf{M} - \mathbf{0} = \mathbf{M}$.

A.3 Matrix Operations: Multiplication

A.3.1 Multiplication of a Matrix by a Number

If a matrix is multiplied by a number (called a *scalar* in matrix algebra), each element in the matrix is simply multiplied by that number. For example

$$2\mathbf{M} = \begin{bmatrix} 4 & 2 & 6 \\ 8 & 12 & 24 \end{bmatrix}$$

A.3.2 Multiplication of a Matrix by another Matrix

Multiplication of two matrices is defined in what appears at first to be a completely illogical way. But we will see that the reason for the definition is precisely because

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of the way in which matrix notation is used for systems of linear relations, especially linear equations. Using **M**, again, and a 3 × 3 matrix $\mathbf{Q} = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, the product $\mathbf{P} = \mathbf{MQ}$, is found as

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 25 \\ 50 & 54 & 88 \end{bmatrix}$$

This comes from

$$\begin{bmatrix} (4+1+9) & (0+1+12) & (8+2+15) \\ (8+6+36) & (0+6+48) & (16+12+60) \end{bmatrix}$$

The rule is: for element p_{ij} in the product, go *across row i* in the matrix on the left (here **M**) and *down column j* in the matrix on the right (here **Q**), multiplying pairs of elements and summing. So, for p_{23} we find, from row 2 of **M** and column 3 of **Q**, $(4)(4) + (6)(2) + (12)(5) = (16 + 12 + 60) = 88$. In general, then, for this example

$$p_{ij} = m_{i1}q_{1j} + m_{i2}q_{2j} + m_{i3}q_{3j} \; (i = 1, 2; \; j = 1, 2, 3)$$

This definition of matrix multiplication means that in order to be *conformable for multiplication* the number of *columns* in the matrix on the left must be the same as the number of *rows* in the matrix on the right. Look again at p_{ij} above; for the three elements in (any) row i of $\mathbf{M}$ – m_{i1}, m_{i2}, and m_{i3} – there must be three "corresponding" elements in (any) column j of $\mathbf{Q}$ – q_{1j}, q_{2j}, and q_{3j}.

The definition of matrix multiplication also means that the product matrix, **P**, will have the same number of rows as **M** and the same number of columns as **Q**. In general,

$$\underset{(m \times n)}{\mathbf{P}} = \underset{(m \times r)}{\mathbf{M}} \; \underset{(r \times n)}{\mathbf{Q}} \tag{A.1}$$

It also means that, in general, order of multiplication makes a difference. In this example, the product the other way around, **QM**, cannot even be found, since there are three columns in **Q** but only two rows in **M**.[2] For that reason, there is language to describe the order of multiplication in a matrix product. For example, in $\mathbf{P} = \mathbf{MQ}$, **M** is said to *premultiply* **Q** (or to multiply **Q** on the left) and, equivalently, **Q** is said to *postmultiply* **M** (or to multiply **M** on the right).

A.3.3 The Identity Matrix

In ordinary algebra, 1 is known as the *identity element for multiplication*, which means that a number remains unchanged when multiplied by it. There is an analogous concept in matrix algebra. An *identity matrix* is one that leaves a matrix unchanged when the matrix is multiplied by it.

[2] Try to carry out the multiplication in the order **QM** to easily see where the trouble arises.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 690.
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If we use $\mathbf{M} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix}$, by what matrix could $\mathbf{M}$ be *post*multiplied so that it remained unchanged? Denote the unknown matrix by $\mathbf{I}$ (this is the standard notation for an identity matrix); we want $\mathbf{MI} = \mathbf{M}$. We know from the rule in (A.1) that $\mathbf{I}$ must be a 3×3 matrix; it needs three rows to be conformable to postmultiply $\mathbf{M}$ and three columns because the product, which will be $\mathbf{M}$ with dimensions 2 by 3, gets its second dimension from the number of columns in $\mathbf{I}$. The reader might try letting $\mathbf{I}$ be a 3×3 matrix with all 1's. It may seem logical but it is wrong. In fact, the only $\mathbf{I}$ for which $\mathbf{MI} = \mathbf{M}$ will be $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The reader should try this and other possibilities, to be convinced that only this matrix will do the job. (Subscripts are often used, as here, to indicate the order of the identity matrix.)

An identity matrix is always square and can be of any size to satisfy the conformability requirement for the particular multiplication operation in which it appears. It has 1's along its *main diagonal*, from upper left to lower right, and 0's everywhere else. We could find another identity matrix by which to *pre*multiply $\mathbf{M}$ so that it remains unchanged. In this case we need the 2×2 identity matrix $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

A.4 Matrix Operations: Transposition

Transposition is a matrix operation for which there is no parallel in ordinary algebra. It plays a useful and important role in certain input–output operations. The *transpose* of an $m \times n$ matrix $\mathbf{M}$, denoted $\mathbf{M}'$, is an $n \times m$ matrix in which row i of $\mathbf{M}$ becomes column i of $\mathbf{M}'$. (Sometimes $\mathbf{M}^t$ or $\mathbf{M}^T$ are used to denote transposition.) For our example

$$\mathbf{M}' = \begin{bmatrix} 2 & 4 \\ 1 & 6 \\ 3 & 12 \end{bmatrix}$$

Notice that the transpose of an n-element column vector (dimensions $n \times 1$) is an n-element row vector (dimensions $1 \times n$).

A useful result, for matrices that are conformable for multiplication, is that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$. The reader can easily see why this is the case by examining a small general example with, say, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$.

A.5 Representation of Linear Equation Systems

Here are two linear equations in two unknowns, x_1 and x_2:

$$\begin{aligned} 2x_1 + x_2 &= 10 \\ 5x_1 + 3x_2 &= 26 \end{aligned} \tag{A.2}$$

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Cambridge, , GBR: Cambridge University Press, 2009. p 691.
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Define **A** as a 2×2 matrix that contains the coefficients multiplying the x's in exactly the order in which they appear, so

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Define a two-element *column* vector, **x**, containing the unknown x's and another two-element column vector, **b**, containing the values on the right-hand sides of the equations exactly in the order in which they appear, namely[3]

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 10 \\ 26 \end{bmatrix}$$

Then, precisely because of the way in which matrix multiplication and matrix equality are defined, the equation system in (A.2) is compactly represented as

$$\mathbf{Ax} = \mathbf{b} \tag{A.3}$$

[Writing out the system represented in (A.3) will show exactly why this is true.]

In ordinary algebra, when we have an equation like $3x = 12$, we "solve" this equation by dividing both sides by 3 – which is the same as multiplying both sides by (1/3), the reciprocal of 3 (sometimes denoted 3^{-1}); multiplication of a number by its reciprocal generates the identity element for multiplication. So, in more detail, we go from $3x = 12$ to $x = 4$ in the logical sequence

$$3x = 12 \Rightarrow (1/3)3x = (1/3)12\ [\text{or } (3^{-1})3x = (3^{-1})12] \Rightarrow (1)x = 4 \Rightarrow x = 4$$

In ordinary algebra the transition from $3x = 12$ to $x = 4$ is virtually immediate. The point here is to set the stage for a parallel approach to systems of linear equations, as in (A.2).

Given the representation in (A.3), it is clear that a way of "solving" this system for the unknowns would be to "divide" both sides by **A**, or, alternatively, multiply both sides by the "reciprocal" of **A**. Parallel to the notation for the reciprocal of a number, this is denoted $\mathbf{A}^{-1}$. If we could find such a matrix, with the property that $(\mathbf{A}^{-1})(\mathbf{A}) = \mathbf{I}$ (the identity element for matrix multiplication), we would proceed in the same way, namely

$$\mathbf{Ax} = \mathbf{b} \Rightarrow (\mathbf{A}^{-1})\mathbf{Ax} = (\mathbf{A}^{-1})\mathbf{b} \Rightarrow \mathbf{Ix} = (\mathbf{A}^{-1})\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

and the values of the unknowns, in **x**, would be found as the matrix operation in which the vector **b** is premultiplied by the matrix $\mathbf{A}^{-1}$, which is usually called the *inverse* of **A**.

[3] The usual convention is to define all vectors as column vectors (as here), so row vectors are formed by transposition.

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A.6 Matrix Operations: Division

In matrix algebra, "division" by a matrix is represented as multiplication by the inverse.[4] Finding inverses can be a very tedious mathematical procedure, but modern computers do it very quickly, even for relatively large matrices. Even though this can easily be done with computer software, we examine a few matrix algebra definitions involving *determinants* and their role in inverses in order to provide a rudimentary understanding of the important concept of a *singular* matrix – one that has no inverse. (The reader uninterested in mathematical details can skip directly to the result on the general definition of an inverse.)

Determinant of a matrix: the 2 × 2 case

A determinant is a number associated with any square matrix. For $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant, $|\mathbf{A}|$, is defined as $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$. Unfortunately, determinants of larger matrices cannot be found by obvious extensions of this simple expression, and additional definitions are needed – specifically minors, cofactors and adjoints.

Minor of an element. The minor of an element a_{ij} in a square matrix $\mathbf{A}$ (denoted m_{ij}) is the determinant of the matrix remaining when row i and column j are removed from $\mathbf{A}$. So the n^2 minors of the elements in $\underset{(n\times n)}{\mathbf{A}}$ will be determinants of $(n-1)\times(n-1)$ matrices.

Cofactor of an element. The cofactor of an element a_{ij} in a square matrix $\mathbf{A}$ (denoted $\mathbf{A}_{ij}$) is defined as $\mathbf{A}_{ij} = (-1)^{i+j}m_{ij}$. When $i+j$ is an even number, $\mathbf{A}_{ij} = m_{ij}$, when $i+j$ is an odd number, $\mathbf{A}_{ij} = -m_{ij}$.

Determinant of a matrix: the general case

For $\underset{(n\times n)}{\mathbf{A}}$, $|\mathbf{A}|$ can be found as

$$\text{(a) } |\mathbf{A}| = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{ij} \text{ (for any } i\text{) or (b) } |\mathbf{A}| = \sum_{i=1}^{n} a_{ij}\mathbf{A}_{ij} \text{ (for any } j\text{)}.$$

In words: $|\mathbf{A}|$ can be found by summing the products of elements and their corresponding cofactors in *any* row [from (a)] or *any* column [from (b)].

Adjoint of a matrix

The adjoint of $\mathbf{A}$ [often denoted (adj $\mathbf{A}$)] is defined as adj $\mathbf{A} = [\mathbf{A}'_{ij}]$. In words: the adjoint is the matrix whose elements are the cofactors of the transpose of $\mathbf{A}$.

[4] In this appendix we will look at inverses for square matrices only. This means that if we are dealing with the coefficient matrix for an equation system, as in (A.2) or (A.3), there are the same number of unknowns as equations in the system. There are more advanced concepts of "pseudo" inverses for nonsquare matrices, but they need not concern us at this point.

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Properties of determinants

1. $|\mathbf{A}| = |\mathbf{A}'|$.
2. If any row or column of $\mathbf{A}$ contains all zeros, $|\mathbf{A}| = 0$.
3. Multiplication of all the elements in any row or column of $\mathbf{A}$ by a constant, k, creates a new matrix whose determinant is $k\,|\mathbf{A}|$.
4. If $\mathbf{A}^*$ is generated from $\mathbf{A}$ by interchanging any two rows or columns of $\mathbf{A}$, $|\mathbf{A}^*| = -|\mathbf{A}|$.
   a. If any two rows or columns in $\mathbf{A}$ are equal, $|\mathbf{A}| = 0$.
   b. If any two rows or columns in $\mathbf{A}$ are proportional, $|\mathbf{A}| = 0$.
5. a. $\sum_{j=1}^{n} a_{ij}\mathbf{A}_{i'j} = 0$ (where $i \neq i'$) and
   b. $\sum_{i=1}^{n} a_{ij}\mathbf{A}_{ij'} = 0$ (where $j \neq j'$).

In words: evaluation of a determinant using *alien cofactors* – elements from row i and cofactors from some other row (which is what makes them *alien*) or elements from column j and cofactors from some other column – always yields a value of zero. This is not difficult to show.

1. Use a_{ij} and $\mathbf{A}_{kj}$ $(i \neq k)$ and write out the alien cofactor expression $\sum_{j=1}^{n} a_{ij}\mathbf{A}_{kj}$.
2. Replace row k in $\mathbf{A}$ by row i; call this matrix $\tilde{\mathbf{A}}$. Then $|\tilde{\mathbf{A}}| = 0$ [from (4a)].
3. Find $|\tilde{\mathbf{A}}|$, which we know to be 0 [from (ii)], by ordinary expansion across its row k; this is $|\tilde{\mathbf{A}}| = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{kj}$. But this is exactly the alien cofactor expression in (i), thus demonstrating (5a) for $i' = k$.

Inverse

The general expression for an inverse builds on the preceding concepts. For the $n \times n$ case, where $\text{adj}\,\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{21} & \cdots & \mathbf{A}_{n1} \\ \mathbf{A}_{12} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{1n} & \mathbf{A}_{2n} & \cdots & \mathbf{A}_{nn} \end{bmatrix}$, form the product

$$\mathbf{A}(\text{adj}\,\mathbf{A}) = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}\mathbf{A}_{1j} & \sum_{j=1}^{n} a_{1j}\mathbf{A}_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}\mathbf{A}_{nj} \\ \sum_{j=1}^{n} a_{2j}\mathbf{A}_{1j} & \sum_{j=1}^{n} a_{2j}\mathbf{A}_{2j} & \cdots & \sum_{j=1}^{n} a_{2j}\mathbf{A}_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{nj}\mathbf{A}_{1j} & \sum_{j=1}^{n} a_{nj}\mathbf{A}_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}\mathbf{A}_{nj} \end{bmatrix}$$

– the reason for doing this will soon be apparent. Each of the on-diagonal elements in this product is $|\mathbf{A}|$, found by cofactor expansions – across each of the rows in turn;

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each off-diagonal element in the product is 0 because it is an expansion by alien cofactors. Therefore

$\mathbf{A}\ (\text{adj}\ \mathbf{A}) = \begin{bmatrix} |\mathbf{A}| & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & |\mathbf{A}| & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}|\,\mathbf{I}_n$, so $\mathbf{A}(1/|\mathbf{A}|)(\text{adj}\ \mathbf{A}) = \mathbf{I}_n$, meaning that $\mathbf{A}^{-1} = \underbrace{(1/|\mathbf{A}|)}_{scalar}\ \underbrace{(\text{adj}\ \mathbf{A})}_{(n\times n)\ matrix}$.

Linear combinations; linear dependence and independence

A more general requirement for nonsingularity of $\mathbf{A}$ involves the concepts of linear dependence and independence. A complete examination of this topic, including the associated vector geometry, is beyond the level of this discussion, but the main ideas are important. We consider only the case of square matrices because our interest is in the inverses of the matrices associated with input–output models.[5] Consider a series of n vectors, either columns or rows; we deal with columns simply for illustration. Let the columns of an $n \times n$ $\mathbf{A}$ matrix be denoted $\mathbf{a}_1^{(c)}, \mathbf{a}_2^{(c)}, \ldots, \mathbf{a}_n^{(c)}$. Multiply each column by a scalar and add, generating another n-element column vector:

$$s_1\mathbf{a}_1^{(c)} + s_2\mathbf{a}_2^{(c)} + \cdots + s_n\mathbf{a}_n^{(c)} = \mathbf{c} \text{ or } \sum_{i=1}^{n} s_i\mathbf{a}_i^{(c)} = \mathbf{c}$$

The vector $\mathbf{c}$ is called a *linear combination* of $\mathbf{a}_1^{(c)}, \mathbf{a}_2^{(c)}, \ldots, \mathbf{a}_n^{(c)}$. If not all the scalars in the linear combination are zero and if $\mathbf{c} = \mathbf{0}$ – that is, $\sum_{i=1}^{n} s_i\mathbf{a}_i^{(c)} = \mathbf{0}$ – $\mathbf{a}_1^{(c)}, \mathbf{a}_2^{(c)}, \ldots, \mathbf{a}_n^{(c)}$ are said to be *linearly dependent*. Using three-element vectors for illustration, suppose $\mathbf{a}_3^{(c)}$ is a linear combination of $\mathbf{a}_1^{(c)}$ and $\mathbf{a}_2^{(c)}$. For example, let $\mathbf{A} = \begin{bmatrix} 1 & 5 & 17 \\ 2 & 4 & 16 \\ 3 & 7 & 27 \end{bmatrix}$; $\mathbf{a}_1^{(c)} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2^{(c)} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$ and $2\mathbf{a}_1^{(c)} + 3\mathbf{a}_2^{(c)} = \mathbf{a}_3^{(c)}$. Then, equivalently,

$$2\mathbf{a}_1^{(c)} + 3\mathbf{a}_2^{(c)} + (-1)\mathbf{a}_3^{(c)} = \mathbf{0}$$

and the vectors $\mathbf{a}_1^{(c)}, \mathbf{a}_2^{(c)}, \mathbf{a}_3^{(c)}$ are (by definition) linearly dependent. Whenever some $\mathbf{a}_i^{(c)}$ can be expressed as a linear combination of the other $(n-1)$ $\mathbf{A}$ vectors, the n vectors $\mathbf{a}_1^{(c)}, \mathbf{a}_2^{(c)}, \ldots, \mathbf{a}_n^{(c)}$ are linearly dependent. The important fact is that if $\underset{(n\times n)}{\mathbf{A}}$ contains linearly

[5] In input–output work, we are usually concerned with finding the inverse of $(\mathbf{I} - \mathbf{A})$. We use a generic "A" matrix in the discussion in this appendix for simplicity of exposition.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 695.
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dependent columns, $\mathbf{A}$ is singular.[6] This provides an additional case in which $|\mathbf{A}| = 0$; it supplements the relatively simpler observations in (2), (4a), and (4b), above. Moreover, all of this holds true if "rows" are substituted for "columns" throughout the discussion; in particular, if $\mathbf{A}$ contains linearly dependent rows, $|\mathbf{A}| = 0$.

On the other hand, if the only scalars for which $\sum_{i=1}^{n} s_i \mathbf{a}_i^{(c)} = 0$ holds are (all) $s_i = 0$, the vectors are termed *linearly independent*. These ideas are used to define the important concept of the *rank* of a matrix, $\rho(\mathbf{A})$. In a nutshell, the rank of $\mathbf{A}$ is the number of linearly independent rows (or columns) in $\mathbf{A}$. And so, if $\rho(\mathbf{A}) = n$, $\mathbf{A}$ is nonsingular. Computer programs find ranks of matrices with very little effort.

One immediate application of these observations can be found with the completely closed input–output model, where $\mathbf{i}'\mathbf{A} = \mathbf{i}'$. As a consequence $\mathbf{i}'(\mathbf{I} - \mathbf{A}) = \mathbf{0}'$ [the rows of $(\mathbf{I} - \mathbf{A})$ are linearly dependent], $|(\mathbf{I} - \mathbf{A})| = 0$, and no Leontief inverse can be found.

Thus $\mathbf{A}^{-1}$ can be found only when $|\mathbf{A}| \neq 0$. This is similar to the problem with "0" in ordinary algebra; you cannot divide by it (the reciprocal of 0, 1/0, is not defined).

The matrix $\mathbf{A}$ from (A.2) is *nonsingular*; namely

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \text{ and } \mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

which the reader can easily check. An example of a singular matrix is $\mathbf{C} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$, where $|\mathbf{C}| = 24 - 24 = 0$ (proportional rows *and* columns). There is no matrix by which $\mathbf{C}$ can be pre- or postmultiplied to generate $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since we have found $\mathbf{A}^{-1}$ for the equations in (A.2), the solution is exactly

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 26 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

The reader can easily check that $x_1 = 4$ and $x_2 = 2$ are the (only) solutions to the two equations in (A.2).

An important fact about inverses is that, for nonsingular matrices $\mathbf{M}$ and $\mathbf{N}$ that are conformable for multiplication, $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$.

A.7 Diagonal Matrices

Identity matrices are examples of *diagonal* matrices. These are always square, with elements on the diagonal from upper left to lower right and zeros elsewhere. In general,

[6] Illustration and proof of this statement is beyond the level of this text. The interested reader should turn to any good book on linear algebra.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 696.
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an $n \times n$ diagonal matrix is

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \cdots & d_n \end{bmatrix}$$

A useful notational device is available for creating a diagonal matrix from a *vector*. Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; then the diagonal matrix with the elements of $\mathbf{x}$ strung out along its main diagonal is denoted by putting a "hat" over the $\mathbf{x}$ (sometimes "⟨" and "⟩" are used to bracket the $\mathbf{x}$), so

$$\hat{\mathbf{x}} = \langle \mathbf{x} \rangle = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

A hat is also used with a square *matrix* to indicate the diagonal matrix formed from the square matrix when all off-diagonal elements are set equal to zero, and an upside down hat is used for the square matrix that is left when all diagonal elements are set equal to zero. For example, using $\mathbf{Q}$ from above,

$$\hat{\mathbf{Q}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \check{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 2 \\ 3 & 4 & 0 \end{bmatrix}$$

One useful fact about diagonal matrices is that the inverse of a diagonal matrix is another diagonal matrix, each of whose elements is just the reciprocal of the original element. For $\hat{\mathbf{x}}$ this means

$$\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 1/x_1 & 0 & 0 \\ 0 & 1/x_2 & 0 \\ 0 & 0 & 1/x_3 \end{bmatrix}$$

and the reader can easily check that in this case

$$\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} = \hat{\mathbf{x}}^{-1}\hat{\mathbf{x}} = \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice also that transposition of a diagonal matrix leaves the matrix unchanged; $\hat{\mathbf{x}}' = \hat{\mathbf{x}}$.

When a diagonal matrix, $\mathbf{D}$, *post*multiplies another matrix, $\mathbf{M}$, the jth element in $\mathbf{D}$, d_j, multiplies all of the elements in the jth *column* of $\mathbf{M}$, and when a diagonal matrix *pre*multiplies $\mathbf{M}$, d_j multiplies all of the elements in the jth *row* of $\mathbf{M}$.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 697.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=731

For example,

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} 2d_1 & d_2 & 3d_3 \\ 4d_1 & 6d_2 & 12d_3 \end{bmatrix}$$

and

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} = \begin{bmatrix} 2d_1 & d_1 & 3d_1 \\ 4d_2 & 6d_2 & 12d_2 \end{bmatrix}$$

Putting the facts about inverses of diagonal matrices together with these observations about pre- and postmultiplication by a diagonal matrix, we see that postmultiplying $\mathbf{M}$ by $\mathbf{D}^{-1}$ will *divide* each element in column j of $\mathbf{M}$ by d_j, and premultiplying $\mathbf{M}$ by $\mathbf{D}^{-1}$ will *divide* each element in row j of $\mathbf{M}$ by d_j.[7]

A.8 Summation Vectors

If $\underset{(m\times n)}{\mathbf{M}}$ is postmultiplied by an n-element column vector of 1's, the results will be an m-element column vector containing the *row sums* of $\mathbf{M}$. If $\mathbf{M}$ is premultiplied by an m-element row vector of 1's, the result will be an n-element row vector containing the *column sums* of $\mathbf{M}$. For example,

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 22 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 15 \end{bmatrix}$$

Usually, a column vector of 1's is denoted by $\mathbf{i}$, and so a corresponding row vector is $\mathbf{i}'$ (sometimes $\mathbf{1}$ or $\mathbf{e}$ is used in place of $\mathbf{i}$). These are *summation vectors*.

A.9 Matrix Inequalities

A more exact characterization of vectors and matrices is often needed for more advanced matrix algebra statements when inequalities are involved. Using vectors as an example, $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x}$ is "non-negative," meaning $x_i \geq 0$ for all i; note that this allows $\mathbf{x} = \mathbf{0}$), $\mathbf{x} > \mathbf{0}$ ($\mathbf{x}$ is "semipositive," meaning $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$; that is, at least one $x_i > 0$) and $\mathbf{x} \gg \mathbf{0}$ ($\mathbf{x}$ is "positive," meaning $x_i > 0$ for all i).[8] The definition of "semipositive" is needed for cases in which $\mathbf{x} = \mathbf{0}$ must be ruled out. The same comparisons can apply to matrices. Also, the same notation can be used to compare any pair of vectors or matrices with the same dimensions – $\mathbf{x} \geq \mathbf{y}$, $\mathbf{x} > \mathbf{y}$, and $\mathbf{x} \gg \mathbf{y}$, and so forth.

[7] This is particularly useful in defining direct input coefficients matrices (technical coefficients matrices) in input–output models.
[8] Alternative notations have been used (for example, in Lancaster, 1968, p. 250, and Takayama, 1985, p. 368). We follow the notation used in Dietzenbacher (1988 and many subsequent publications).

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 698.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=732

A.10 Partitioned Matrices

Often it is useful to divide a matrix into *submatrices*, especially if there is some logical reason to distinguish some rows and columns from others.[9] This is known as *partitioning* the matrix; the submatrices are sometimes separated by dashed or dotted lines. For example, we might create four submatrices from a 4×4 matrix $\mathbf{A}$, as

$$\underset{(4\times 4)}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

In the discussion of linear combinations (section A.6), we viewed $\mathbf{A}$ as composed of a series of column vectors, $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{(c)} & \mathbf{a}_2^{(c)} & \cdots & \mathbf{a}_n^{(c)} \end{bmatrix}$. It can equally well be thought of as a "stack" of row vectors, $\mathbf{a}_i^{(r)}$ – namely, $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{(r)} \\ \mathbf{a}_2^{(r)} \\ \vdots \\ \mathbf{a}_n^{(r)} \end{bmatrix}$.

A.10.1 Multiplying Partitioned Matrices

If matrices are partitioned so that submatrices are conformable for multiplication, then products of partitioned matrices can be found as products of these submatrices. For example, suppose that in conjunction with $\mathbf{A}$, above, we have

$$\underset{(4\times 3)}{\mathbf{B}} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

Then

$$\underset{(4\times 3)}{\mathbf{AB}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

(The reader can check that all conformability requirements for addition and for multiplication are met.)

[9] An example is in the representation of interregional or multiregional input–output models.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 699.
http://site.ebrary.com/lib/mitlibraries/Doc?id=10329730&ppg=733

A.10.2 The Inverse of a Partitioned Matrix

Inverses of partitioned matrices play an important role in many input–output representations. Given a partitioned $n \times n$ matrix $\mathbf{A} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{E}} & \underset{[p\times(n-p)]}{\mathbf{F}} \\ \underset{[(n-p)\times p]}{\mathbf{G}} & \underset{[(n-p)\times(n-p)]}{\mathbf{H}} \end{bmatrix}$ (note that $\mathbf{E}$ and $\mathbf{H}$ are square), elements of the inverse can be similarly partitioned as $\mathbf{A}^{-1} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{S}} & \underset{[p\times(n-p)]}{\mathbf{T}} \\ \underset{[(n-p)\times p]}{\mathbf{U}} & \underset{[(n-p)\times(n-p)]}{\mathbf{V}} \end{bmatrix}$. Notice that submatrices in corresponding locations in the original matrix and the inverse have the same dimensions. This means that

$$\begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{T} \\ \mathbf{U} & \mathbf{V} \end{bmatrix} = \mathbf{I} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{I}} & \underset{[p\times(n-p)]}{\mathbf{0}} \\ \underset{[(n-p)\times p]}{\mathbf{0}} & \underset{[(n-p)\times(n-p)]}{\mathbf{I}} \end{bmatrix}$$

That is, the product (the identity matrix) can also be partitioned similarly. This matrix statement can be expanded into four *matrix* equations, using the usual rules for matrix multiplication and matrix equality. These matrix equations are

$$\begin{array}{ll} (1)\ \mathbf{ES} + \mathbf{FU} = \mathbf{I} & (3)\ \mathbf{ET} + \mathbf{FV} = \mathbf{0} \\ (2)\ \mathbf{GS} + \mathbf{HU} = \mathbf{0} & (4)\ \mathbf{GT} + \mathbf{HV} = \mathbf{I} \end{array} \tag{A.4}$$

(The reader can easily check that all matrices are conformable for the multiplications and additions in which they are involved.)

Assume that $\mathbf{E}^{-1}$ can be found; then (1) yields $\mathbf{S} = \mathbf{E}^{-1}(\mathbf{I} - \mathbf{FU})$. Putting this into (2), after considerable rearrangement, gives $\mathbf{U} = -(\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1}\mathbf{GE}^{-1}$. The important fact is that $\mathbf{U}$ is expressed as a function of only the known matrices $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$; and once $\mathbf{U}$ is found, it can be substituted back into the expression for $\mathbf{S}$. Similarly, equations (3) and (4) can be solved to yield $\mathbf{T} = -\mathbf{E}^{-1}\mathbf{FV}$ and $\mathbf{V} = (\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1}$. As with the first pair of equations, $\mathbf{V}$ is a function of known matrices only, and once $\mathbf{V}$ is found, it can be used to find $\mathbf{T}$. Collecting these results,

$$\begin{array}{ll} \mathbf{S} = \mathbf{E}^{-1}(\mathbf{I} - \mathbf{FU}) & \mathbf{T} = -\mathbf{E}^{-1}\mathbf{FV} \\ \mathbf{U} = -\mathbf{VGE}^{-1} & \mathbf{V} = (\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1} \end{array} \tag{A.5}$$

In this way, the inverse of an $n \times n$ matrix is found from the inverses of two smaller matrices – $\underset{(p\times p)}{\mathbf{E}}$ and $\underset{[(n-p)\times(n-p)]}{\mathbf{V}}$ – along with a number of matrix multiplications.

Miller, Ronald E.; Blair, Peter D.. Input-Output Analysis : Foundations and Extensions.
Cambridge, , GBR: Cambridge University Press, 2009. p 700.
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An alternative set of results can be derived if one begins with the assumption that $\mathbf{H}^{-1}$ is known. These are

$$\begin{aligned} \mathbf{S} &= (\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G})^{-1} & \mathbf{T} &= -\mathbf{S}\mathbf{F}\mathbf{H}^{-1} \\ \mathbf{U} &= -\mathbf{H}^{-1}\mathbf{G}\mathbf{S} & \mathbf{V} &= \mathbf{H}^{-1}(\mathbf{I} - \mathbf{G}\mathbf{T}) \end{aligned} \tag{A.6}$$

Again, inverses of two (different) smaller matrices are required – $\underset{(p\times p)}{\mathbf{S}}$ and $\underset{[(n-p)\times(n-p)]}{\mathbf{H}}$.

For $\mathbf{A}$ matrices with particular structures the solution via (A.5) or (A.6) may be particularly simple. Here are several alternatives that arise in input–output models.

1. If $\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix}$ then, using either (A.5) or (A.6), it is easily established that $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix}$; only the two smaller inverses, $\mathbf{E}^{-1}$ and $\mathbf{H}^{-1}$, are needed.
2. In the even more special case when $\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$.
3. If $\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{G} & \mathbf{I} \end{bmatrix}$, then $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ -\mathbf{G}\mathbf{E}^{-1} & \mathbf{I} \end{bmatrix}$.

The interested reader can easily construct additional variations on these special cases.

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